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On probabilistic linguistic term set operations

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Abstract. In a recent work (Wang et al. 2020), a partial order $\preceq$, a join operation $\sqcup$ and a meet operation $\sqcap$ of probabilistic linguistic term sets (PLTSs) were introduced and it was proved that $L_1 \sqcap L_2 \preceq L_1 \preceq L_1 \sqcup L_2$ and $L_1 \sqcap L_2 \preceq L_2 \preceq L_1 \sqcup L_2$. In this paper, we demonstrate that its join and meet operations are not satisfy the above requirement. To satisfy this requirement, we modify its join and meet operations. Moreover, we define a negation operation of PLTSs based on the partial order $\preceq$. The combinations of the proposed negation, the modified join and meet operations yield a bounded, distributive lattice over PLTSs. Meanwhile, we also define a new join operation and a new meet operation which, together with the negation operation, yield a bounded De Morgan over PLTSs.

Keywords: Probabilistic linguistic term sets, operations, orders, lattices

1. Introduction

1.1. Related works

Many mathematical modelings have been presented recently to deal with randomness, fuzziness, vagueness and uncertainty of decision environment. To deal with the complexity of decision making, combining different theories together has emerged as an important trend, such as probabilistic rough sets [1, 2], rough graphs [3–5], fuzzy soft graphs [6, 7] and Hesitant fuzzy linguistic term sets [8, 9].

Probabilistic linguistic term sets (PLTSs) was introduced by Pang et al. [10] as a combination of probability theory and linguistic term sets. In recent years, PLTSs have been successful applied in several applications, such as supplier evaluation [11], sustainability evaluation [12], shelter selection [13], and online product selection [14]. Especially, evidence theory is used widely as a group decision making method.

In many theoretical and application studies of PLTSs, operations play an important role. Several operations were defined based on different models, such as the symbolic linguistic model [10, 15, 16], and semantic linguistic model [17, 18]. Some authors used evidence theory to define PLTSs operations [19, 20]. Very recently, Wang et al. [21] introduced several operations of PLTSs based on the stochastic order of PLTSs. A reasonable operation of PLTSs is closely related to the order of PLTSs. For example, a join operation $\sqcup$ and a meet operation $\sqcap$ should satisfy $L_1 \sqcap L_2 \preceq L_1 \preceq L_1 \sqcup L_2$ and $L_1 \sqcap L_2 \preceq L_2 \preceq L_1 \sqcup L_2$ for the order $\preceq$ of PLTSs. However, we argue that the join operation and meet operation in [21] do not satisfy the above requirement. In a word, Theorem 4 in [21] is incorrect.

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1.2. Motivation of our research

The motivation of this paper comes from three aspects as following:

1) As discussed above, the study on probabilistic linguistic term set operations is a research topic with important theoretical and practical effect. That is why it has been an issue worth of current research, and hence it deserves our further pursuit.
2) The join operation $\sqcup$ and meet operation $\sqcap$ are two fundamental operations. $L_1 \sqcap L_2 \preceq L_1 \preceq L_1 \sqcup L_2$ and $L_1 \sqcap L_2 \preceq L_2 \preceq L_1 \sqcup L_2$ are their basic requirements. If the join operation and meet operation do not satisfy the basic requirements, then we need to reconsider these two fundamental operations.
3) The negation $\neg$ is another fundamental operation. We cannot ignore the negation operation in the area of probabilistic linguistic term set operations, and hence it deserves our further pursuit.

Based on aforementioned consideration, in this paper, as a supplement of these topics from the theoretical point of view, we first modify join and meet operations in Wang et al [21], and then introduce the negation operation which have not been studied in details in the literature. Additionally, we introduce a new pair of join and meet operations. The main results of the paper are lattice properties of these probabilistic linguistic term set operations. Comparatively, our proposed operations have the following advantages.

1) The proposed operations are satisfactorily consistent with Wang et al.'s partial order $\preceq$ of PLTSs. This partial order is reasonable from the point of the classical stochastic order in probabilistic theory.
2) The combinations of the proposed negation, join and meet operations yield De Morgan lattices over PLTSs. This means, the proposed operations are interconnected by an algebraic structure.

1.3. Framework of the paper

The rest of this article is organized as follows. Section 2 recalls necessary preliminaries regarding PLTSs. In Section 3, we modify Wang et al.'s join and meet operations, define some new operations and study their lattice properties. In Section 4, an example is given to show the effectiveness of our operators applied in decision making. The article is concluded in Section 5.

2. Preliminaries

Linguistic variables are effective to evaluate qualitative information of objects [22, 23]. In general, we use a linguistic term set to contain all possible values of a linguistic variable:

$$S = \{s_0, s_1, \cdots, s_\lambda\} \quad (1)$$

where $\lambda + 1$ is the granularity of S, s_α is generated by a predefined syntactic rule and restricted by a fuzzy set.

Definition 1. [10]. Let $S = \{s_0, s_1, \cdots, s_\lambda\}$ be an linguistic term set. A probabilistic linguistic term set L on S is a subset of S in which each linguistic term is associated with its probability:

$$L = \left\{ s_k(p_k) | s_k \in S, p_k \geq 0, k = 0, 1, \cdots, \lambda, \sum_{k=0}^{\lambda} p_k \leq 1 \right\} \quad (2)$$

where p_k is the probability of s_k being the real value of the linguistic variable.

Let us denote by $\mathbb{L}_S$ the set of all probabilistic linguistic term set on the linguistic term set S.

Remark 1. $\varepsilon_S = 1 - \sum_{k=0}^{\lambda} p_k$ is called ignorance, which are useful in the incomplete evaluations. In this paper, we assume that $\varepsilon_S = 0$.

For a probabilistic linguistic term set L, its cumulative distribution function is defined as:

$$F_L(x) = \sum_{\alpha \leq x} \Pr(\{s_\alpha\}) = \sum_{\alpha \leq x} p_\alpha, x \in \mathbb{R} \quad (3)$$

where $\mathbb{R}$ is the set of real numbers.

Definition 2. [21]. The partial ordering of $\mathbb{L}_S$ induced by the stochastic order is defined as follows $\forall L_1, L_2 \in \mathbb{L}_S$

$$L_1 \preceq L_2 \Leftrightarrow F_{L_1}(x) \geq F_{L_2}(x), \forall x \in \mathbb{R}. \quad (4)$$

They [21] also define the join and meet operations as follow:

$$F_{L_1 \sqcup L_2}(x) = \max\{F_{L_1}(x), F_{L_2}(x)\}, \forall x \in \mathbb{R}; \quad (5)$$

$$F_{L_1 \sqcap L_2}(x) = \min\{F_{L_1}(x), F_{L_2}(x)\}, \forall x \in \mathbb{R}. \quad (6)$$

Obviously, $\forall x \in \mathbb{R}$

$$\max\{F_{L_1}(x), F_{L_2}(x)\} \geq F_{L_1}(x) \geq \min\{F_{L_1}(x), F_{L_2}(x)\}. \quad (7)$$

By the definition of $\preceq$, we have

$$L_1 \sqcup L_2 \preceq L_1 \preceq L_1 \sqcap L_2. \quad (8)$$

This is obviously unreasonable. Theorem 4 in [21] is incorrect.

3. Probabilistic linguistic term set operations

3.1. Join and meet operations

We first modify Wang et al.'s join and meet operations as follow:

$$F_{L_1 \cup L_2}(x) = \min\{F_{L_1}(x), F_{L_2}(x)\}, \forall x \in \mathbb{R}; \quad (9)$$

$$F_{L_1 \cap L_2}(x) = \max\{F_{L_1}(x), F_{L_2}(x)\}, \forall x \in \mathbb{R}. \quad (10)$$

Theorem 1. *For any* $L_1, L_2, L_3 \in \mathbb{L}_S$,

(i) $L_1 \cup L_2$ and $L_1 \cap L_2$ are PLTSs;

(ii)

$$\begin{aligned} L_1 \cap L_2 \preceq L_1 \preceq L_1 \cup L_2, \\ L_1 \cap L_2 \preceq L_2 \preceq L_1 \cup L_2; \end{aligned} \quad (11)$$

(iii)

$$\begin{aligned} L_1 \cap (L_2 \cup L_3) = (L_1 \cap L_2) \cup (L_1 \cap L_3), \\ L_1 \cup (L_2 \cap L_3) = (L_1 \cup L_2) \cap (L_1 \cup L_3). \end{aligned}$$

Proof. Omitted. □

3.2. Negation operations

First, we define a negation operation on $\mathbb{L}_S$ based on $\preceq$.

Definition 3. A function $\neg : \mathbb{L}_S \to \mathbb{L}_S$ is a negation if

(i) $\neg L^{\top} = L_{\perp}, \neg L_{\perp} = L^{\top}$ where $L^{\top} = \{s_{\lambda}(1)\}$ and $L^{\top} = \{s_0(1)\}$;

(ii) $\neg$ is an inverted order mapping, i.e.,

$$L_1 \preceq L_2 \Rightarrow \neg L_2 \preceq \neg L_1. \quad (12)$$

In addition, a negation $\neg$ satisfies:

(iii) the *continuity* property if $\neg$ is a continuous function;

(iv) the *involutivity* property if $\neg$ is an involution, i.e., for any $L \in \mathbb{L}_S$

$$\neg\big(\neg L\big) = L. \quad (13)$$

Theorem 2. *For any* $L = \Big\{s_k(p_k)|s_k \in S, p_k \geq 0, k = 0, 1, \cdots, \lambda, \sum_{k=0}^{\lambda} p_k = 1\Big\} \in \mathbb{L}_S$, *define*

$$\neg L = \{s_0(p_{\lambda}), s_1(p_{\lambda} - 1), \cdots, s_{\lambda}(p_0)\}. \quad (14)$$

Then it is a negation of L. Moreover, it is an involution.

Proof. (i) and (iv) are clear.

(ii) Let $L_1 = \{s_k(p_k)|k = 0, 1, \cdots, \lambda\}$ and $L_2 = \{s_k(q_k)|k = 0, 1, \cdots, \lambda\}$. If $L_1 \preceq L_2$, i.e., for any $t \geq 0$,

$$\sum_{j=0}^{t} p_j \geq \sum_{j=0}^{t} q_j.$$

Then

$$1 - \sum_{j=0}^{t} p_j \leq 1 - \sum_{j=0}^{t} q_j.$$

By $\sum\limits_{j=0}^{\lambda} p_j = \sum\limits_{j=0}^{\lambda} q_j = 1$, we have for any $a \leq \lambda$,

$$\sum_{j=a}^{\lambda} p_j \leq \sum_{j=a}^{\lambda} q_j.$$

This means $F_{\neg L_1}(x) \leq F_{\neg L_2}(x), \forall x \in \mathbb{R}$. We thus get $\neg L_1 \succeq \neg L_2$. □

Example 1. Given a linguistic term set $S = \{s_0, s_1, s_2\}$. Consider two PLTSs

$$L_1 = (s_0(0.1), s_1(0.2), s_2(0.7)), \quad (15)$$

$$L_2 = (s_0(0.2), s_1(0.5), s_2(0.3)). \quad (16)$$

Their cumulative distribution functions respectively are

$$F_{L_1}(x) = \begin{cases} 0, & x < 0, \\ 0.1, & 0 \leq x < 1, \\ 0.3, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \quad (17)$$

$$F_{L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.2, & 0 \le x < 1, \\ 0.7, & 1 \le x < 2, \\ 1, & x \ge 3. \end{cases} \tag{18}$$

Clearly, $L_1 \succeq L_2$ because $F_{L_1}(x) \le F_{L_2}(x), \forall x \in \mathbb{R}$.

By the definition of $\neg$ in Equation 14, we have

$$\neg L_1 = (s_0(0.7), s_1(0.2), s_2(0.1)), \tag{19}$$

$$\neg L_2 = (s_0(0.3), s_1(0.5), s_2(0.2)). \tag{20}$$

Then their cumulative distribution functions respectively are

$$F_{\neg L_1}(x) = \begin{cases} 0, & x < 0, \\ 0.7, & 0 \le x < 1, \\ 0.9, & 1 \le x < 2, \\ 1, & x \ge 3. \end{cases} \tag{21}$$

$$F_{\neg L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.3, & 0 \le x < 1, \\ 0.8, & 1 \le x < 2, \\ 1, & x \ge 3. \end{cases} \tag{22}$$

Clearly, $\neg L_1 \preceq \neg L_2$ because $F_{\neg L_1}(x) \ge F_{\neg L_2}(x), \forall x \in \mathbb{R}$.

Now we consider the continuity of the negation operation $\neg$. First, we need a distance measure for $\mathbb{L}_S$.

Definition 4. A function $d : \mathbb{L}_S \times \mathbb{L}_S \to [0, 1]$ is called a distance measure of $\mathbb{L}_S$, if for any $L_1, L_2, L_3 \in \mathbb{L}_S$,

(D1) $0 \le d(L_1, L_2) \le 1$,
(D2) $d(L_1, L_2) = 1$ if and only if $L_1 = L_2$,
(D3) $d(L_1, L_2) = d(L_2, L_1)$,
(D4) $d(L_1, L_2) + d(L_2, L_3) \ge d(L_1, L_3)$.

Theorem 3. *For any $L_1, L_2 \in \mathbb{L}_S$, the following function*

$$d(L_1, L_2) = \sup_{x\in\mathbb{R}} |F_{L_1}(x) - F_{L_2}(x)| \tag{23}$$

is a distance measure of $\mathbb{L}_S$.

Proof. (D1)-(D3) are clear.

(D4) For any $L_1, L_2, L_3 \in \mathbb{L}_S$

$$\begin{aligned} & d(L_1, L_3) \\ &= \sup_{x\in\mathbb{R}} |F_{L_1}(x) - F_{L_3}(x)| \\ &= \sup_{x\in\mathbb{R}} |F_{L_1}(x) - F_{L_2}(x) + F_{L_2}(x) - F_{L_3}(x)| \\ &\le \sup_{x\in\mathbb{R}} \left(|F_{L_1}(x) - F_{L_2}(x)| + |F_{L_2}(x) - F_{L_3}(x)|\right) \\ &\le \sup_{x\in\mathbb{R}} |F_{L_1}(x) - F_{L_2}(x)| + \sup_{x\in\mathbb{R}} |F_{L_2}(x) - F_{L_3}(x)| \\ &= d(L_1, L_2) + d(L_2, L_3). \end{aligned}$$

□

Theorem 4. *If $L_1 \preceq L_2 \preceq L_3$ then $d(L_1, L_3) \ge d(L_1, L_2)$ and $d(L_1, L_3) \ge d(L_2, L_3)$.*

Proof. Omitted. □

Theorem 5. *For any $L_1, L_2 \in \mathbb{L}_S$,*

$$d(L_1, L_2) = d(\neg L_2, \neg L_1)$$

Proof. Let

$$L_1 = \left\{s_0(p_0), s_1(p_1), \cdots, s_\lambda(p_\lambda)\right\},$$

$$L_2 = \left\{s_0(q_0), s_1(q_1), \cdots, s_\lambda(q_\lambda)\right\}.$$

Then

$$\neg L_1 = \left\{s_0(p_\lambda), s_1(p_{\lambda-1}), \cdots, s_\lambda(p_0)\right\},$$

$$\neg L_2 = \left\{s_0(q_\lambda), s_1(q_{\lambda-1}), \cdots, s_\lambda(q_0)\right\}.$$

By the definition of the diatance d, we have

$$\begin{aligned} d(L_1, L_2) &= \sup_{t\ge 0} \Big| \sum_{j=0}^{t} p_j - \sum_{j=0}^{t} q_j \Big| \\ &= \sup_{t\ge 0} \Big| (1 - \sum_{j=0}^{t} p_j) - (1 - \sum_{j=0}^{t} q_j) \Big| \\ &= \sup_{t\ge 0} \Big| \sum_{j=t+1}^{\lambda} p_j - \sum_{j=t+1}^{\lambda} q_j \Big| \\ &= d(\neg L_1, \neg L_2). \end{aligned}$$

□

Corollary 1. *The negation operation $\neg$ in Equation 14 is continuous in the distance of Equation 23.*

Example 2. Consider three PLTSs as follows:

$$L_1 = \Big\{s_0(0.4), s_1(0.5), s_3(0.1)\Big\},$$

$$L_2 = \Big\{s_0(0.5), s_1(0.4), s_3(0.1)\Big\},$$

$$L_3 = \Big\{s_0(0.3), s_1(0.5), s_3(0.2)\Big\}.$$

Then their cumulative distribution functions respectively are

$$F_{L_1}(x) = \begin{cases} 0, & x < 0, \\ 0.4, & 0 \leq x < 1, \\ 0.9, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{24}$$

$$F_{L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.5, & 0 \leq x < 1, \\ 0.9, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{25}$$

$$F_{L_3}(x) = \begin{cases} 0, & x < 0, \\ 0.3, & 0 \leq x < 1, \\ 0.8, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{26}$$

It is clear that $F_{L_3}(x) \leq F_{L_1}(x) \leq F_{L_2}(x), \forall x \in \mathbb{R}$,i.e., $L_2 \preceq L_1 \preceq L_3$.

$$d(L_1, L_2) = \sup_x |F_{L_1}(x) - F_{L_2}(x)| = 0.1,$$
$$d(L_1, L_3) = \sup_x |F_{L_1}(x) - F_{L_3}(x)| = 0.1,$$
$$d(L_2, L_3) = \sup_x |F_{L_2}(x) - F_{L_3}(x)| = 0.2.$$

Clearly, we have $d(L_1, L_2) < d(L_2, L_3)$ and $d(L_1, L_3) < d(L_2, L_3)$.

3.3. Lattice properties

The negation $\neg$, join $\cup$ and meet $\cap$ operations on $\mathbb{L}_S$ have the following properties.

Theorem 6. *For any $L_1, L_2, L_3 \in \mathbb{L}_S$, the following properties hold*:

(1) $L_1 \cap L_1 = L_1$, $L_1 \cup L_1 = L_1$,
(2) $L_1 \cap L_2 = L_2 \cap L_1$, $L_1 \cup L_2 = L_2 \cup L_1$,
(3) $L_1 \cup L^{\top} = L^{\top}$ and $L_1 \cap L^{\top} = L_1$,
(4) $L_1 \cap L_{\perp} = L_{\perp}$ and $L_1 \cup L_{\perp} = L_1$,
(5) $(L_1 \cap L_2) \cap L_3 = L_1 \cap (L_2 \cap L_3)$,
$(L_1 \cup L_2) \cup L_3 = L_1 \cup (L_2 \cup L_3)$,
(6) $\neg(L_1 \cap L_2) = \neg L_1 \cup \neg L_2$,
$\neg(L_1 \cup L_2) = \neg L_1 \cap \neg L_2$.

Proof. (1)-(5) are clear. We only give the proof of (6). Let

$$L_1 = \Big\{s_0(p_0), s_1(p_1), \cdots, s_\lambda(p_\lambda)\Big\},$$

$$L_2 = \Big\{s_0(q_0), s_1(q_1), \cdots, s_\lambda(q_\lambda)\Big\}.$$

$$F_{L_1 \cap L_2}(x) = \max(F_{L_1}(x), F_{L_2}(x)), \forall x \in \mathbb{R}$$

Then

$$L_1 \cap L_2 = \Big\{s_0(g(0)), s_1(g(1)), \cdots, s_\lambda(g(\lambda))\Big\}$$

where

$$g(y) = F_{L_1 \cap L_2}(y) - F_{L_1 \cap L_2}(y-1), y = 0, 1, \cdots, \lambda.$$

Then

$$\neg(L_1 \cap L_2) = \Big\{s_0(f(0)), s_1(f(1)), \cdots, s_\lambda(f(\lambda))\Big\}$$

where

$$\begin{aligned} f(x) &= g(\lambda - x) \\ &= F_{L_1 \cap L_2}(\lambda - x) - F_{L_1 \cap L_2}(\lambda - x - 1), \end{aligned}$$

for $x = 0, 1, \cdots, \lambda$.

Because

$$\neg L_1 = \Big\{s_0(p_\lambda), s_1(p_{\lambda-1}), \cdots, s_\lambda(p_0)\Big\},$$

$$\neg L_2 = \Big\{s_0(q_\lambda), s_1(q_{\lambda-1}), \cdots, s_\lambda(q_0)\Big\}.$$

For any $x = 0, 1, \cdots, \lambda$

$$\begin{aligned} &F_{\neg L_1 \cup \neg L_2}(x) \\ &= \min(F_{\neg L_1}(x), F_{\neg L_2}(x)\Big) \\ &= \min\Big(\sum_{j=\lambda-x}^{\lambda} p_j, \sum_{j=\lambda-x}^{\lambda} q_j\Big) \\ &= \min\Big(1 - \sum_{j=0}^{\lambda-x-1} p_j, 1 - \sum_{j=0}^{\lambda-x-1} q_j\Big) \\ &= 1 - \max\Big(\sum_{j=0}^{\lambda-x-1} p_j, \sum_{j=0}^{\lambda-x-1} q_j\Big) \end{aligned}$$

$$= 1 - \max\left(F_{L_1}(\lambda - x - 1), F_{L_2}(\lambda - x - 1)\right)$$

Then

$$\neg L_1 \cup \neg L_2 = \left\{s_0(h(0)), s_1(h(1)), \cdots, s_\lambda(h(\lambda))\right\}$$

where

$$\begin{aligned} h(x) &= \left(1 - \max\left(F_{L_1}(\lambda - x - 1), F_{L_2}(\lambda - x - 1)\right)\right) \\ &\quad - \left(1 - \max\left(F_{L_1}(\lambda - x), F_{L_2}(\lambda - x)\right)\right) \\ &= \max\left(F_{L_1}(\lambda - x), F_{L_2}(\lambda - x)\right) \\ &\quad - \max\left(F_{L_1}(\lambda - x - 1), F_{L_2}(\lambda - x - 1)\right) \\ &= F_{L_1 \cap L_2}(\lambda - x) - F_{L_1 \cap L_2}(\lambda - x - 1) \\ &= f(x) \end{aligned}$$

for $x = 0, 1, \cdots, \lambda$.

We thus get $\neg(L_1 \cap L_2) = \neg L_1 \cup \neg L_2$. We can prove $\neg(L_1 \cup L_2) = \neg L_1 \cap \neg L_2$ in a similar manner. □

Example 3. Consider two PLTSs as follows:

$$L_1 = \left\{s_0(0.7), s_1(0.1), s_3(0.2)\right\},$$

$$L_2 = \left\{s_0(0.6), s_1(0.3), s_3(0.1)\right\}.$$

Then their cumulative distribution functions respectively are

$$F_{L_1}(x) = \begin{cases} 0, & x < 0, \\ 0.7, & 0 \leq x < 1, \\ 0.8, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{27}$$

$$F_{L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.6, & 0 \leq x < 1, \\ 0.9, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{28}$$

Then

$$F_{L_1 \cup L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.6, & 0 \leq x < 1, \\ 0.8, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{29}$$

$$F_{L_1 \cap L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.7, & 0 \leq x < 1, \\ 0.9, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{30}$$

Thus

$$L_1 \cup L_2 = \left\{s_0(0.6), s_1(0.2), s_3(0.2)\right\}, \tag{31}$$

$$L_1 \cap L_2 = \left\{s_0(0.7), s_1(0.2), s_3(0.1)\right\}. \tag{32}$$

Then

$$\neg(L_1 \cup L_2) = \left\{s_0(0.2), s_1(0.2), s_3(0.6)\right\}, \tag{33}$$

$$\neg(L_1 \cap L_2) = \left\{s_0(0.1), s_1(0.2), s_3(0.7)\right\}. \tag{34}$$

By the concept of $\neg$, then

$$\neg L_1 = \left\{s_0(0.2), s_1(0.1), s_3(0.7)\right\}, \tag{35}$$

$$\neg L_2 = \left\{s_0(0.1), s_1(0.3), s_3(0.6)\right\}. \tag{36}$$

And their cumulative distribution functions respectively are

$$F_{\neg L_1}(x) = \begin{cases} 0, & x < 0, \\ 0.2, & 0 \leq x < 1, \\ 0.3, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{37}$$

$$F_{\neg L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.1, & 0 \leq x < 1, \\ 0.4, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{38}$$

Then

$$F_{\neg L_1 \cap \neg L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.2, & 0 \leq x < 1, \\ 0.4, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{39}$$

$$F_{\neg L_1 \cup \neg L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.1, & 0 \leq x < 1, \\ 0.3, & 1 \leq x < 2, \\ 1, & x \geq 3. \end{cases} \tag{40}$$

Thus

$$\neg L_1 \cap \neg L_2 = \left\{s_0(0.2), s_1(0.2), s_3(0.6)\right\}, \tag{41}$$

$$\neg L_1 \cup \neg L_2 = \left\{s_0(0.1), s_1(0.2), s_3(0.7)\right\}. \tag{42}$$

Clearly, $\neg(L_1 \cup L_2) = \neg L_1 \cap \neg L_2$ and $\neg(L_1 \cap L_2) = \neg L_1 \cup \neg L_2$.

Theorem 7. $(\mathbb{L}_S, \cap, \cup, \neg, L_\perp, L^\top)$ *is a bounded De Morgan lattice.*

Theorem 8. $(\mathbb{L}_S, \cap, \cup, \neg, L_\perp, L^\top)$ *is a distributive lattice.*

We give a connection between $\cap, \cup$ and Wang et al's order $\preceq$ in Definition 2.

Corollary 2. *Let $L_1, L_2 \in \mathbb{L}_S$, the following are equivalent:*

(1) $L_1 \preceq L_2$;
(2) $L_1 \cap L_2 = L_1$;
(3) $L_1 \cup L_2 = L_2$.

Note that, this relationship between Wang et al's $\sqcap, \sqcup$ and $\preceq$ is not violated .

3.4. New join and meet operations

We define a new pair of join and meet operations on $\mathbb{L}_S$.

Definition 5. For any two PLTSs L_1 and L_2,

$$L_1 = \Big\{s_0(p_0), s_1(p_1), \cdots, s_\lambda(p_\lambda)\Big\},$$
$$L_2 = \Big\{s_0(q_0), s_1(q_1), \cdots, s_\lambda(q_\lambda)\Big\}.$$

the join and meet of L_1 and L_2 are, respectively, given by

$$L_1 \vee L_2 = \Big(s_0(p_0 \wedge q_0), \cdots, s_{\lambda-1}(p_{\lambda-1} \wedge q_{\lambda-1}), s_\lambda\big(1 - \big(\sum_{j=0}^{\lambda-1} p_j \wedge q_j\big)\big)\Big), \tag{43}$$

$$L_1 \wedge L_2 = \Big(s_0\big(1 - \big(\sum_{j=1}^{\lambda} p_j \wedge q_j\big)\big), s_1(p_1 \wedge q_1), \cdots, s_\lambda(p_\lambda \wedge q_\lambda)\Big). \tag{44}$$

Lemma 1. *For any two PLTSs L_1 and L_2,*

(1) $L_1 \wedge L_2$ and $L_1 \vee L_2$ are weighting vectors,
(2) $L_2 \preceq L_1 \vee L_2$ and $L_1 \preceq L_1 \vee L_2$,
(3) $L_1 \wedge L_2 \preceq L_1$ and $L_1 \wedge L_2 \preceq L_2$.

Proof. (1) First, we have $p_0 \wedge q_0, \cdots, p_{\lambda-1} \wedge q_{\lambda-1} \in [0, 1]$ and $0 \le \big(\sum_{j=0}^{\lambda-1} p_j \wedge q_j\big) \le \big(\sum_{j=1}^{\lambda-1} p_j\big) \le 1$, then $1 - \big(\sum_{j=0}^{\lambda-1} p_j \wedge q_j\big) \in [0, 1]$. Second, $p_0 \wedge q_0 + \cdots + p_{\lambda-1} \wedge q_{\lambda-1} + 1 - \big(\sum_{j=0}^{\lambda-1} p_j \wedge q_j\big) = \big(\sum_{j=0}^{\lambda-1} p_j \wedge q_j\big) + 1 - \big(\sum_{j=0}^{\lambda-1} p_j \wedge q_j\big) = 1.$
Thus $L_1 \wedge L_2$ is a PLTS. Similarly, we can show that $L_1 \vee L_2$ is a PLTS.

(2) First, we give the proof of $L_1 \vee L_2 \succeq L_1$. For any $j = 0, 1, \cdots, \lambda - 1$, $p_j \wedge q_j \le p_j$, then for any $x = 0, 1, \cdots, \lambda - 1$,

$$\sum_{j=1}^{x} p_j \wedge q_j \le \sum_{j=1}^{x} p_j,$$

We thus get $F_{L_1 \vee L_2}(x) \le F_{L_1}(x)$, i.e., $L_1 \vee L_2 \succeq L_1$. Similarly, we can get $L_1 \vee L_2 \succeq L_2$.

(3) Second, we give the proof of $L_1 \wedge L_2 \preceq L_1$. For any $j = 1, 2, \cdots, n - 1$, $p_j \wedge q_j \le q_j$, then for any $x = 1, \cdots, \lambda$,

$$\sum_{j=x}^{\lambda} p_j \wedge q_j \le \sum_{j=1}^{x} p_j,$$

This means $\neg(L_1 \wedge L_2) \succeq \neg(L_1)$. By the involutivity of $\neg$, we have $L_1 \wedge L_2 \preceq L_1$.

Similarly, we can get $L_1 \wedge L_2 \preceq L_2$. □

Moreover, the negation $\neg$, conjunction $\wedge$ and disjunction $\vee$ operations have the following properties.

Theorem 9. *For any three PLTSs*

$$L_1 = \Big\{s_0(p_0), s_1(p_1), \cdots, s_\lambda(p_\lambda)\Big\},$$
$$L_2 = \Big\{s_0(q_0), s_1(q_1), \cdots, s_\lambda(q_\lambda)\Big\},$$
$$L_3 = \Big\{s_0(q_0), s_1(q_1), \cdots, s_\lambda(r_\lambda)\Big\}.$$

the following properties hold:

(1) $L_1 \wedge L_1 = L_1$, $L_1 \vee L_1 = L_1$,
(2) $L_1 \wedge L_2 = L_2 \wedge L_1$, $L_1 \vee L_2 = L_2 \vee L_1$,
(3) $L_1 \vee L^\top = L^\top$ and $L_1 \wedge L^\top = \Big\{s_0(1 - p_\lambda), s_\lambda(p_\lambda)\Big\}$,
(4) $L_1 \wedge L_\perp = L_\perp$ and $L_1 \vee L_\perp = \Big\{s_0(p_0), s_\lambda(1 - p_0)\Big\}$,

(5) $(L_1 \wedge L_2) \wedge L_3 = L_1 \wedge (L_2 \wedge L_3)$,
$(L_1 \vee L_2) \vee L_3 = L_1 \vee (L_2 \vee L_3)$,
(6) $\neg(L_1 \wedge L_2) = \neg L_1 \vee \neg L_2$,
$\neg(L_1 \vee L_2) = \neg L_1 \wedge \neg \vee L_2$.

Proof. (1)-(5) are clear. We only give the proof of (6):

$$\begin{aligned}
&\neg L_1 \vee \neg L_2 \\
=&\Big\{s_0(p_\lambda), s_1(p_{\lambda-1}), \cdots, s_\lambda(p_0)\Big\} \\
&\vee \Big\{s_0(q_\lambda), s_1(q_{\lambda-1}), \cdots, s_\lambda(q_0)\Big\} \\
=&\Big\{s_0(p_\lambda \wedge q_\lambda), \cdots, s_{\lambda-1}(p_1 \wedge q_1), \\
&s_\lambda(1 - \big(\sum_{j=1}^{\lambda} p_j \wedge q_j)\big)\Big\} \\
=&\neg(L_1 \wedge L_2).
\end{aligned}$$

$$\begin{aligned}
&\neg L_1 \wedge \neg L_2 \\
=&\Big\{s_0(p_\lambda), s_1(p_{\lambda-1}), \cdots, s_\lambda(p_0)\Big\} \\
&\wedge \Big\{s_0(q_\lambda), s_1(q_{\lambda-1}), \cdots, s_\lambda(q_0)\Big\} \\
=&\Big\{s_0(1 - \big(\sum_{j=0}^{\lambda-1} p_j \wedge q_j)\big), s_{\lambda-1}(p_{\lambda-1} \wedge q_{\lambda-1}), \\
&\cdots, s_\lambda(p_0 \wedge q_0)\Big\} \\
=&\neg(L_1 \vee L_2).
\end{aligned}$$

□

Example 4. Consider two PLTSs as follows:

$$L_1 = \Big\{s_0(0.7), s_1(0.1), s_3(0.2)\Big\},$$
$$L_2 = \Big\{s_0(0.6), s_1(0.3), s_3(0.1)\Big\}.$$

Their cumulative distribution functions respectively are

$$F_{L_1}(x) = \begin{cases} 0, & x < 0, \\ 0.7, & 0 \le x < 1, \\ 0.8, & 1 \le x < 2, \\ 1, & x \ge 3. \end{cases} \tag{45}$$

$$F_{L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.6, & 0 \le x < 1, \\ 0.9, & 1 \le x < 2, \\ 1, & x \ge 3. \end{cases} \tag{46}$$

Clearly, $F_{L_1}(x) \nleq F_{L_2}(x)$ and $F_{L_2}(x) \nleq F_{L_1}(x)$. Thus $L_1 \not\succeq L_2$ and $L_1 \not\preceq L_2$.

By the concepts of $\wedge$ and $\vee$ in Definition 5, then

$$L_1 \vee L_2 = \Big\{s_0(0.6), s_1(0.1), s_3(0.3)\Big\},$$
$$L_1 \wedge L_2 = \Big\{s_0(0.8), s_1(0.1), s_3(0.1)\Big\}.$$

Their cumulative distribution functions respectively are

$$F_{L_1 \vee L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.6, & 0 \le x < 1, \\ 0.7, & 1 \le x < 2, \\ 1, & x \ge 3. \end{cases} \tag{47}$$

$$F_{L_1 \wedge L_2}(x) = \begin{cases} 0, & x < 0, \\ 0.8, & 0 \le x < 1, \\ 0.9, & 1 \le x < 2, \\ 1, & x \ge 3. \end{cases} \tag{48}$$

Then

$$F_{L_1 \vee L_2}(x) \le F_{L_1}(x) \le F_{L_1 \vee L_2}(x), \forall x \in \mathbb{R}, \tag{49}$$
$$F_{L_1 \vee L_2}(x) \le F_{L_2}(x) \le F_{L_1 \vee L_2}(x), \forall x \in \mathbb{R}. \tag{50}$$

Thus we have

$$L_1 \vee L_2 \succeq L_1 \succeq L_1 \wedge L_2, \tag{51}$$
$$L_1 \vee L_2 \succeq L_2 \succeq L_1 \wedge L_2. \tag{52}$$

By the concept of $\neg$, then

$$\neg L_1 = \Big\{s_0(0.2), s_1(0.1), s_3(0.7)\Big\},$$
$$\neg L_2 = \Big\{s_0(0.1), s_1(0.3), s_3(0.6)\Big\}.$$

Moreover, we have

$$(\neg L_1 \wedge \neg L_2) = \Big\{s_0(0.3), s_1(0.1), s_3(0.6)\Big\},$$
$$(\neg L_1 \vee \neg L_2) = \Big\{s_0(0.1), s_1(0.1), s_3(0.8)\Big\},$$

and

$$\neg(L_1 \vee L_2) = \Big\{s_0(0.3), s_1(0.1), s_3(0.6)\Big\},$$

$$\neg(L_1 \wedge L_2) = \left\{s_0(0.1), s_1(0.1), s_3(0.8)\right\}, \tag{53}$$

Clearly, $(\neg L_1 \wedge \neg L_2) = \neg(L_1 \vee L_2)$ and $(\neg L_1 \vee \neg L_2) = \neg(L_1 \wedge L_2)$.

Theorem 10. *$(\mathbb{L}_S, \wedge, \vee, \neg, L_\perp, L^\top)$ is a bounded De Morgan lattice.*

We also give a connection between $\wedge$, $\vee$ and Wang et al's order $\preceq$ in Definition 2.

Corollary 3. *Let $L_1, L_2 \in \mathbb{L}_S$, the following are equivalent:*

(1) $L_1 \preceq L_2$;
(2) $L_1 \wedge L_2 = L_1$;
(3) $L_1 \vee L_2 = L_2$.

Theorem 11. *$(\mathbb{L}_S, \wedge, \vee, \neg, L_\perp, L^\top)$ is not a distributive lattice.*

Proof. Consider three PLTSs as follows:

$$L_1 = \left\{s_0(0.4), s_1(0.3), s_3(0.3)\right\},$$

$$L_2 = \left\{s_0(0.5), s_1(0.1), s_3(0.4)\right\},$$

$$L_3 = \left\{s_0(0.3), s_1(0.5), s_3(0.2)\right\}.$$

Then

$$\begin{aligned}
&(L_1 \wedge L_2) \vee (L_1 \wedge L_3)\\
&= \big((s_0(0.4), s_1(0.3), s_3(0.3))\\
&\qquad \wedge (s_0(0.3), s_1(0.5), s_3(0.2))\big)\\
&\vee\big((s_0(0.7), s_1(0.1), s_3(0.2))\\
&\qquad \wedge (s_0(0.6), s_1(0.3), s_3(0.1))\big)\\
&= (s_0(0.5), s_1(0.3), s_3(0.2))\\
&\qquad \vee (s_0(0.8), s_1(0.1), s_3(0.1))\\
&= (s_0(0.5), s_1(0.1), s_3(0.4)).
\end{aligned}$$

and

$$\begin{aligned}
&L_1 \wedge (L_2 \vee L_3)\\
&= (s_0(0.4), s_1(0.3), s_3(0.3))\\
&\wedge\big((s_0(0.5), s_1(0.1), s_3(0.4))\\
&\qquad \vee (s_0(0.3), s_1(0.5), s_3(0.2))\big)\\
&= (s_0(0.4), s_1(0.3), s_3(0.3)\\
&\qquad \wedge (s_0(0.7), s_1(0.1), s_3(0.2)\\
&= (s_0(0.7), s_1(0.1), s_3(0.2).
\end{aligned}$$

Thus $(L_1 \wedge L_2) \vee (L_1 \wedge L_3) \neq L_1 \wedge (L_2 \vee L_3)$. □

From above theorems, we can see that $(\mathbb{L}_S, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ does not satisfy the condition of distributivity. So we consider the modularity condition which is weaker than the distributivity condition.

Theorem 12. *$(\mathbb{L}_S, \wedge, \vee, \neg, L_\perp, L^\top)$ does not satisfy the condition of modularity.*

Proof. If it satisfies the condition of modularity. Then $(L_1 \wedge (L_2) \vee (L_1 \wedge (L_3) = L_1 \wedge (L_2 \vee (L_1 \wedge L_3))$ for any L_1, L_2 and L_3.

Here, we present a counter-example on the modularity. Consider L_1, L_2 and L_3 in the above Theorem, then

$$\begin{aligned}
&L_1 \wedge (L_2 \vee (L_1 \wedge L_3))\\
&= (s_0(0.4), s_1(0.3), s_3(0.3))\\
&\quad \wedge \Big((s_0(0.5), s_1(0.1), s_3(0.4))\\
&\qquad \vee \big((s_0(0.4), s_1(0.3), s_3(0.3))\\
&\qquad\quad \wedge (s_0(0.3), s_1(0.5), s_3(0.2))\big)\Big)\\
&= (s_0(0.4), s_1(0.3), s_3(0.3))\\
&\quad \wedge \Big((s_0(0.5), s_1(0.1), s_3(0.4))\\
&\qquad\quad \vee \big(s_0(0.5), s_1(0.3), s_3(0.2)\big)\Big)\\
&= (s_0(0.4), s_1(0.3), s_3(0.3))\\
&\qquad \wedge (s_0(0.5), s_1(0.1), s_3(0.4))\\
&= (s_0(0.6), s_1(0.1), s_3(0.3)).
\end{aligned}$$

Thus $(L_1 \wedge (L_2) \vee (L_1 \wedge (L_3) \neq L_1 \wedge (L_2 \vee (L_1 \wedge L_3))$ for this example of L_1, L_2 and L_3. □

4. Multi-attribute group decision making

In this section, our operators are applied to decision making with probabilistic linguistic information.

Let $X = \{x_1, x_2, ..., x_n\}$ be the set of n alternatives, and $A = \{a_1, a_2, ..., a_m\}$ be the set of m attributes and $S = \{s_0, s_1, ..., s_\lambda\}$ the linguistic term set. Assume that $D = \{d_1, d_2, ..., d_p\}$ is the set of decision makers and $R^{(i)} = (s^{(i)}_{(a_jk)})_{(p\times n)}$ is their probabilistic linguistic decision matrix, where each $s^{(i)}_{(a_{jk})}$ is a PLTSs on S and represents the linguistic assessment of the alternative $x_k \in X$ with respect to the attributes $a_j \in A$ obtained by the decision maker $d_i \in D$.

Table 1
The probabilistic linguistic decision matrix provided by the first decision maker

	a_1	a_2
x_1	$\{s_0(0.2), s_1(0.1), s_2(0.7)\}$	$\{s_0(0.3), s_1(0.1), s_2(0.6)\}$
x_2	$\{s_0(0.1), s_1(0.3), s_2(0.6)\}$	$\{s_0(0.1), s_1(0.4), s_2(0.5)\}$
x_3	$\{s_0(0.1), s_1(0.1), s_2(0.8)\}$	$\{s_0(0.1), s_1(0.3), s_2(0.6)\}$

Applying the min ∩ operator on PLTSs, our selection method of the alternatives is given as follows:

Seep 1: We utilize the min operator to aggregate all $s^{(i)}_{(a_{jk})}(i = 1, 2, ..., p)$ for each decision maker $d_i \in D$,

$$s_{(a_{jk})} = s^{(1)}_{(a_jk)} \cap s^{(2)}_{(a_jk)} \cap \cdots \cap s^{(p)}_{(a_jk)}. \quad (54)$$

Then we get a probabilistic linguistic decision matrix $R = (s_{(a_jk)})_{(p\times n)}$.

Seep 2: We utilize the min operator to aggregate all $s_{(a_{jk})}(k = 1, 2, ..., m)$ for each attribute $a_i \in A$,

$$s_j = s_{(a_j1)} \cap s_{(a_j2)} \cap \cdots \cap s_{(a_j3)}. \quad (55)$$

Then we get a probabilistic linguistic decision vector $V = (s_j)_(n)$.

Seep 3: We calculate the distance between each alternative and the positive ideal solution (PIS) of alternatives by using Equation 23.

Seep 4: Rank all the alternatives. Obviously, the smaller the distance, the better the alternative.

Note that in this case, the decision makers are pessimistic. If they are optimistic, then we have max ∪ operator instead of min ∩ operator in Equations 54 and 55.

In the following, we illustrate the operation of the decision making with an example.

Suppose that there are three possible products $x_i(i = 1, 2, 3)$ to be evaluate. It is necessary to compare these products so as to select the best one as well as order the taking into account two attributes: a_1 quality perspective and a_2 service perspective. The three decision makers utilize the following LTS:

$$S = \{s_0 = low, s_1 = medium, s_2 = high\}$$

to evaluate the products $x_i(i = 1, 2, 3)$ by means of PLTSs. The probabilistic linguistic decision matrix of the decision makers are given in Tables 1–3.

Step 1. By using Equation 54, the probabilistic linguistic decision matrix of the group is shown in Table 4.

Table 2
The probabilistic linguistic decision matrix provided by the second decision maker

	a_1	a_2
x_1	$\{s_0(0.1), s_1(0.1), s_2(0.8)\}$	$\{s_0(0.3), s_1(0.1), s_2(0.6)\}$
x_2	$\{s_0(0.1), s_1(0.2), s_2(0.7)\}$	$\{s_0(0.4), s_1(0), s_2(0.6)\}$
x_3	$\{s_0(0.2), s_1(0.2), s_2(0.6)\}$	$\{s_0(0.1), s_1(0), s_2(0.9)\}$

Table 3
The probabilistic linguistic decision matrix provided by the third decision maker

	a_1	a_2
x_1	$\{s_0(0.1), s_1(0.2), s_2(0.7)\}$	$\{s_0(0.2), s_1(0.2), s_2(0.6)\}$
x_2	$\{s_0(0), s_1(0.1), s_2(0.9)\}$	$\{s_0(0.2), s_1(0.1), s_2(0.7)\}$
x_3	$\{s_0(0.2), s_1(0.2), s_2(0.6)\}$	$\{s_0(0), s_1(0.4), s_2(0.6)\}$

Table 4
The probabilistic linguistic decision matrix of the group

	a_1	a_2
x_1	$\{s_0(0.2), s_1(0.1), s_2(0.7)\}$	$\{s_0(0.3), s_1(0.1), s_2(0.6)\}$
x_2	$\{s_0(0.1), s_1(0.3), s_2(0.6)\}$	$\{s_0(0.4), s_1(0.1), s_2(0.5)\}$
x_3	$\{s_0(0.2), s_1(0.2), s_2(0.6)\}$	$\{s_0(0.1), s_1(0.3), s_2(0.6)\}$

Step 2. By using Equation 55, the probabilistic linguistic decision vector of the group is

$$\begin{pmatrix} \{s_0(0.3), s_1(0.1), s_2(0.6)\} \\ \{s_0(0.4), s_1(0.1), s_2(0.5)\} \\ \{s_0(0.2), s_1(0.2), s_2(0.6)\} \end{pmatrix}.$$

Step 3. The positive ideal solution of alternatives $L^* = \{s_0(0), s_1(0), s_2(1)\}$, we calculate the distance between each alternative and the positive ideal solution,

$$d(x_1, L^*) = 0.7, d(x_2, L^*) = 0.6, d(x_3, L^*) = 0.8.$$

Step 4. Rank the alternatives $x_i(i = 1, 2, 3)$ according to the distances $d(x_i, L^*)(i = 1, 2, 3)$: $x_2 \succ x_1 \succ x_3$ and thus, the best alternative is x_2.

5. Conclusion

In this paper, we improved the theories introduced by Wang et al. [21]. Our contributions can be summarized as follow.

1. We modified Wang et al.'s join and meet operations to satisfy the requirement $L_1 \cap L_2 \preceq L_1 \preceq L_1 \cup L_2$ and $L_1 \cap L_2 \preceq L_2 \preceq L_1 \cup L_2$.

2. We defined an involution negation operation and a distance of PLTSs based on Wang et al.'s partial order $\preceq$. The proposed negation operation is continuous in the proposed distance.
3. We demonstrate that $(\mathbb{L}_S, \cap, \cup, \neg, L_\perp, L^\top)$ is a bounded De Morgan lattice and also is a distributive lattice.
4. We demonstrate that $(\mathbb{L}_S, \wedge, \vee, \neg, L_\perp, L^\top)$ is a bounded De Morgan lattice, but it is not a distributive lattice. Moreover, it does not satisfy the condition of modularity.

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