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Novel entropy and distance measures of linguistic interval-valued q-Rung orthopair fuzzy sets

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Abstract. Entropy is an important tool to describe the degree of uncertainty of fuzzy sets. In this study, we first define a new entropy and distance measure in the linguistic q-Rung orthopair fuzzy (LIVqROF) environment, and verify its correctness and rationality. Secondly, in the LIVqROF environment, the new entropy formula is effectively applied to the multi-attribute decision making (MADM) with unknown attribute weights, which provides a new idea for solving the MADM problems. Finally, the feasibility and effectiveness of the proposed method are verified by a numerical example.

Keywords: Entropy measure, distance measure, linguistic interval-valued q-Rung orthopair fuzzy set, group decision making

1. Introduction

Zadeh [1] proposed the fuzzy set (FS), which only has the membership degree, and it is difficult to describe the negative degree of the decision maker to the evaluation information. Atanassov [2, 3] proposed intuitionistic fuzzy set (IFS) on the basis of fuzzy set, and creatively added non-membership degree to the fuzzy sets. However, intuitionistic fuzzy numbers can only be used to describe the evaluation information in the form of real numbers. Due to the complexity of the decision-making environment, the decision making information is often uncertain. Sometimes it is very difficult to express the membership and non-membership degree with real numbers. Therefore, Atanassov [4] also proposed interval intuitionistic fuzzy set (IVIFS), which membership and non-membership are interval numbers rather than real numbers. In real life, the form of linguistic terms is often used to qualitatively evaluate information. Zadeh [5] proposed the fuzzy linguistic method, and Zhang [6] defined the concept of linguistic intuitionistic fuzzy set (LIF). Similar to intuitionistic fuzzy sets, linguistic intuitionistic fuzzy sets are also limited by the representation range. Inspired by the concepts of Pythagorean fuzzy set (PFS) and q-Rung orthopair fuzzy set (qROFS) introduced by Yager in [7] and [8], Garg [9] proposed linguistic Pythagorean fuzzy set (LPFS) and Lin [10] developed the concept of linguistic q-Rung orthopair fuzzy set (LqROFS). Linguistic interval intuitionistic fuzzy numbers can better deal with the uncertainty and fuzziness of decision information, and can truly reflect the objective world. Garg proposed linguistic interval intuitionistic fuzzy set (LIVIFS) and linguistic interval Pythagorean fuzzy set (LIVPFS) in [11] and [12], respectively. Khan [13] established linguistic interval q-Rung orthopair fuzzy set (LIVqROFS).

Information measure plays an important role in fuzzy set theory, and many researchers have explored it from different perspectives. In order to deal with the

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fuzziness measurement between FSs, De Luca and termini [14] gave an axiomatic definition of fuzzy entropy based on Shannon [15]. Since then, Hooda et al. [16, 17], Mishra et al. [18–21], Pal [22] have studied various entropy measures of FSs. Burillo and Bustance [23] defined the intuitionistic fuzzy entropy to measure the hesitation degree of intuitionistic fuzzy sets. Liu [24] presented the interval intuitionistic fuzzy entropy.Li [25, 26] gave the study of slope entropy and fractional slope entropy. Zhang and Jiang [27], Zhang [28], Wei [29], and Wei and Zhang [30] introduced the entropy measure of IVIFSs and applied it in solving MADM problems. Rani [31] gave a new entropy measure of IVIFS, Kumar et al. [32] gave a new Pythagorean fuzzy entropy, Sonia et al. [33] gave a entropy measures for interval valued intuitionistic fuzzy soft set and Ohlan [34] presented a distance measure on IVIFS. Inspired by their research, this paper defines an entropy measure and distance measure in LIVqROF environment. The relationship between the proposed entropy measure and distance measure are studied.

The rest of this paper is arranged as follows. In Section 2, we introduce some definitions and operation rules on IVIFSs and LIVqROFSs. In Section 3, the axiomatic definitions of entropy and distance in LIVqROF environment are given, a new entropy and distance are proposed, and their rationality and relevance are proved. In Section 4, a multi-attribute decision making method is proposed in LIVqROF environment by using the proposed entropy. A numerical example is used to verify the effectiveness and rationality of the method.

2. Preliminary

The linguistic assessment scale is the basis of linguistic decision making. In [35], Bordogna et al. defined an additive linguistic scale.

Definition 2.1. Let

$$S = \{s_\alpha | \alpha = 0, 1, \cdots, t\}$$

be an additive linguistic assessment scale with a linguistic term subscript non-negative integer, and let S satisfy the following conditions:

(1) The set is ordered: $s_\alpha < s_\beta$ if $\alpha < \beta$.

(2) There is a negative operator: $neg(s_\alpha) = s_\beta$ such that $\alpha + \beta = t$.

where t is an even number and s_α refers to linguistic terms.

After that, Xu [36] extended the discrete term set S to a continuous term set $\bar{S} = \{s_\alpha | \alpha \in [0, \tau]\}$, where $\tau(\tau > t)$ is a sufficiently large natural number. If $s_\alpha \in S$, then we call s_α the original linguistic term, otherwise, we call s_α the virtual linguistic term.

Combining the concept of intuitionistic fuzzy sets [2] with the definition of linguistic terms [35], Zhang [6] first proposed the concept of linguistic intuitionistic fuzzy sets(LIFSs), which is defined as follows.

Definition 2.2. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe discourse and $\bar{S} = \{s_\alpha | \alpha \in [0, \tau]\}$ be a continuous linguistic term set with a positive integer τ. A LIFS on the set X is defined as

$$I = \{(x_i, s_\theta(x_i), s_\sigma(x_i)) | x_i \in X\}, \quad (1)$$

where $s_\theta : X \to \bar{S}$ denotes the degree of linguistic membership and $s_\sigma : X \to \bar{S}$ denotes the degree of linguistic nonmembership of the element $x_i \in X$ to the set I, respectively, with the condition that $0 \leq \theta + \sigma \leq \tau$. The degree of linguistic indeterminacy is given as $s_\pi(x_i) = s_{\tau-\theta-\sigma}$.

For convenience, the pairs of $(s_\theta(x_i), s_\sigma(x_i))$ are called as linguistic intuitionistic fuzzy value(LIFV) or linguistic intuitionistic fuzzy number(LIFN).

On the basis of LIFSs and the concept of Pythagorean fuzzy sets [7], Garg [9] proposed the concept of linguistic Pythagorean fuzzy sets(LPFSs), which is defined as follows.

Definition 2.3. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe discourse and $\bar{S} = \{s_\alpha | \alpha \in [0, \tau]\}$ be a continuous linguistic term set with a positive integer τ. A LPFS on the set X is defined with the form

$$P = \{(x_i, s_\theta(x_i), s_\sigma(x_i)) | x_i \in X\}, \quad (2)$$

where $s_\theta(x_i), s_\sigma(x_i) \in \bar{S}$ stand for the linguistic membership degree and linguistic nonmembership degree of the element x_i to P, respectively, with the condition that $0 \leq \theta^2 + \sigma^2 \leq \tau^2$. The degree of linguistic indeterminacy is given as: $s_\pi(x_i) = s_{\sqrt{\tau^2-\theta^2-\sigma^2}}$.

Similarly, the pairs of $(s_\theta(x_i), s_\sigma(x_i))$ are called as linguistic Pythagorean fuzzy value(LPFV) or linguistic Pythagorean fuzzy number(LPFN).

On the basis of LIFSs, LPFSs and q-rung orthopair fuzzy sets [8], Lin [10] proposed linguistic q-rung orthopair fuzzy sets(LqROFSs), which is defined as follows.

Definition 2.4. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe discourse and $\bar{S} = \{s_\alpha | \alpha \in [0, \tau]\}$ be a continuous linguistic term set with a positive integer τ. Then the form of a LqROFS on X is defined as

$$L = \{(x_i, s_\theta(x_i), s_\sigma(x_i))|x_i \in X\}, \tag{3}$$

where $s_\theta(x_i), s_\sigma(x_i) \in \bar{S}$ stand for the linguistic membership degree and linguistic nonmembership degree of the element x_i to L, respectively, with the condition that $0 \leq \theta^q + \sigma^q \leq \tau^q$. The degree of linguistic indeterminacy is given as: $s_\pi(x_i) = s_{\sqrt[q]{\tau^q - \theta^q - \sigma^q}}$.

Similarly, the orthopair of $(s_\theta(x_i), s_\sigma(x_i))$ is called as linguistic q-rung orthopair fuzzy value(LqROFV) or linguistic q-rung orthopair fuzzy number(LqROFN).

Obviously, when $q = 1$ and $q = 2$, the two special cases of LqROFSs are LIFSs and LPFSs, respectively.

In [4], Atanassov and Gargov proposed the concept of interval-valued intuitionistic fuzzy sets(IVIFSs), which is defined as follows.

Definition 2.5. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe discourse. Then the form of a IVIFS on Y is defined as

$$\hat{I} = \{(x_i, \hat{\theta}(x_i), \hat{\sigma}(x_i))|x_i \in X\}, \tag{4}$$

where $\hat{\theta}(x_i) = [\hat{\theta}_L(x_i), \hat{\theta}_U(x_i)], \hat{\sigma}(x_i) = [\hat{\sigma}_L(x_i), \hat{\sigma}_U(x_i)]$ are all subsets of $[0, 1]$ and are said to be the membership degree and nonmembership degree of the element x_i to $\hat{I}$, respectively, with the condition that $0 \leq \hat{\theta}_U + \hat{\sigma}_U \leq 1$. The degree of indeterminacy is given as: $\hat{\pi}_{\hat{I}}(x_i) = [\hat{\pi}_L(x_i), \hat{\pi}_U(x_i)] = [1 - \hat{\theta}_U(x_i) - \hat{\sigma}_U(x_i), 1 - \hat{\theta}_L(x_i) - \hat{\sigma}_L(x_i)]$.

For convenience, the pairs of $(\hat{\theta}(x_i), \hat{\sigma}(x_i))$ are called as interval-valued intuitionistic fuzzy value(IVIFV) or interval-valued intuitionistic fuzzy number(IVIFN).

If $\hat{\theta}_L(x_i) = \hat{\theta}_U(x_i)$ and $\hat{\sigma}_L(x_i) = \hat{\sigma}_U(x_i)$, then the given IVIFSs can be reduced to intuitionistic fuzzy sets(IFSs).

In [11, 37], Garg, Liu et al. defined the concept of linguistic interval-valued intuitionistic fuzzy sets(LIVIFSs). In [12], Garg proposed the concept of linguistic interval-valued Pythagorean fuzzy sets(LIVPFSs). On this basis, Khan et al. [13] further proposed linguistic interval-valued q-rung orthopair fuzzy sets(LIVqROFSs), which is defined as follows.

Definition 2.6. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe discourse and $\bar{S} = \{s_\alpha | \alpha \in [0, \tau]\}$ be a continuous linguistic term set with a positive integer τ. Then the form of a LIVqROFS on X is defined as

$$\hat{L} = \{(x_i, s_{\hat{\theta}}(x_i), s_{\hat{\sigma}}(x_i))|x_i \in X\}, \tag{5}$$

where $s_{\hat{\theta}}(x_i) = [s_{\hat{\theta}_L}(x_i), s_{\hat{\theta}_U}(x_i)], s_{\hat{\sigma}}(x_i) = [s_{\hat{\sigma}_L}(x_i), s_{\hat{\sigma}_U}(x_i)]$ are all subsets of $[s_0, s_\tau]$ and are said to be the membership degree and nonmembership degree of the element x_i to $\hat{L}$, respectively, with the condition that $0 \leq \hat{\theta}_U^q + \hat{\sigma}_U^q \leq \tau^q$. The degree of indeterminacy is given as:

$$s_{\hat{\pi}}(x_i) = [s_{\hat{\pi}_L}(x_i), s_{\hat{\pi}_U}(x_i)] = [s_{\sqrt[q]{\tau^q - \hat{\theta}_U^q - \hat{\sigma}_U^q}}, s_{\sqrt[q]{\tau^q - \hat{\theta}_L^q - \hat{\sigma}_L^q}}].$$

Obviously, when $q = 1$ and $q = 2$, the two special cases of LIVqROFSs are LIVIFSs and LIVPFSs, respectively. If $s_{\hat{\theta}_L}(x_i) = s_{\hat{\theta}_U}(x_i)$ and $s_{\hat{\sigma}_L}(x_i) = s_{\hat{\sigma}_U}(x_i)$, then the given LIVqROFSs can be reduced to an ordinary LqROFSs.

For convenience, the pairs of $(s_{\hat{\theta}}(x_i), s_{\hat{\sigma}}(x_i))$ are called as linguistic interval-valued q-rung orthopair fuzzy value (LIVqROFV) or linguistic interval-valued q-rung orthopair fuzzy number (LIVqROFN).

Definition 2.7. [38] Let $\hat{A}_i = (s_{\hat{\theta}_i}, s_{\hat{\sigma}_i}) = ([s_{\hat{\theta}_{iL}}, s_{\hat{\theta}_{iU}}], [s_{\hat{\sigma}_{iL}}, s_{\hat{\sigma}_{iU}}]), i = 1, 2$ be two LIVqROFNs. Then

(1) If $s_{\hat{\theta}_{1L}} = s_{\hat{\theta}_{2L}}, s_{\hat{\theta}_{1U}} = s_{\hat{\theta}_{2U}}, s_{\hat{\sigma}_{1L}} = s_{\hat{\sigma}_{2L}}, s_{\hat{\sigma}_{1U}} = s_{\hat{\sigma}_{2U}}$, then $\hat{A}_1 = \hat{A}_2$;
(2) If $s_{\hat{\theta}_{1L}} \leq s_{\hat{\theta}_{2L}}, s_{\hat{\theta}_{1U}} \leq s_{\hat{\theta}_{2U}}, s_{\hat{\sigma}_{1L}} \geq s_{\hat{\sigma}_{2L}}, s_{\hat{\sigma}_{1U}} \geq s_{\hat{\sigma}_{2U}}$, then $\hat{A}_1 \leq \hat{A}_2$;
(3) Negation of $\hat{A}_1$ is defined as $\hat{A}_1^C = ([s_{\hat{\sigma}_{1L}}, s_{\hat{\sigma}_{1U}}], [s_{\hat{\theta}_{1L}}, s_{\hat{\theta}_{1U}}])$.

In [39], Bustince and Burillo proposed the axiomatic definition of entropy measure for IFSs. Analogously, in [24], Liu, Zheng and Xiong proposed the entropy measure of IVIFSs, which is defined as follows.

Definition 2.8. An entropy measure of $IVIFS(X)$ is a real-valued function $F : IVIFS(X) \rightarrow [0, 1]$, if it satisfies the following conditions:

($F1$) $F(\hat{A}) = 0$ if and only if $\hat{A} = ([1, 1], [0, 0])$ or $\hat{A} = ([0, 0], [1, 1])$ for each $x_i \in X$;
($F2$) $F(\hat{A}) = 1$ if and only if $[\hat{\theta}_L(x_i), \hat{\theta}_U(x_i)] = [\hat{\sigma}_L(x_i), \hat{\sigma}_U(x_i)]$ for each $x_i \in X$;
($F3$) $F(\hat{A}) = F(\hat{A}^C)$, where $\hat{A}^C = \{(x_i, \hat{\sigma}(x_i), \hat{\theta}(x_i))|x_i \in X\}$;
($F4$) $F(\hat{A}_1) \leq F(\hat{A}_2)$ if $\hat{A}_1 \leq \hat{A}_2$ (i.e. $\hat{\theta}_{1L}(x_i) \leq \hat{\theta}_{2L}(x_i), \hat{\theta}_{1U}(x_i) \leq \hat{\theta}_{2U}(x_i), \hat{\sigma}_{1L}(x_i) \geq$

$\hat{\sigma}_{2L}(x_i), \hat{\sigma}_{1U}(x_i) \geq \hat{\sigma}_{2U}(x_i))$ when $\hat{\theta}_{2L}(x_i) \leq \hat{\sigma}_{2L}(x_i)$ and $\hat{\theta}_{2U}(x_i) \leq \hat{\sigma}_{2U}(x_i)$ for each $x_i \in X$, or $\hat{A}_2 \leq \hat{A}_1$ when $\hat{\theta}_{2L}(x_i) \geq \hat{\sigma}_{2L}(x_i)$ and $\hat{\theta}_{2U}(x_i) \geq \hat{\sigma}_{2U}(x_i)$.

In [40], Dü ğenci proposed the distance measure on IVIFS(X), defined as follows.

Definition 2.9. Let $\hat{A}_1, \hat{A}_2 \in IVIFS(X)$, a mapping $D : IVIFS(X) \times IVIFS(X) \to [0, 1]$ is called a distance measure between $\hat{A}_1$ and $\hat{A}_2$ if $D(\hat{A}_1, \hat{A}_2)$ satisfies the following properties:

- $(D1)$ $0 \leq D(\hat{A}_1, \hat{A}_2) \leq 1$;
- $(D2)$ $D(\hat{A}_1, \hat{A}_2) = 0$ if $\hat{A}_1 = \hat{A}_2$;
- $(D3)$ $D(\hat{A}_1, \hat{A}_2) = D(\hat{A}_2, \hat{A}_1)$;
- $(D4)$ If $\hat{A}_1 \leq \hat{A}_2 \leq \hat{A}_3$, $\hat{A}_1, \hat{A}_2, \hat{A}_3 \in IVIFS(X)$, then $D(\hat{A}_1, \hat{A}_2) \leq D(\hat{A}_1, \hat{A}_3)$ and $D(\hat{A}_2, \hat{A}_3) \leq D(\hat{A}_1, \hat{A}_3)$.

3. Entropy and distance measures for LIVqROFSs

Pal [41] proposed the exponential entropy measure of a fuzzy set A ($A \in F(X) = \{(x_i, \mu(x_i))|x_i \in X\}$) as

$$E(A) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} (\mu(x_i)e^{1-\mu(x_i)} + (1-\mu(x_i))e^{\mu(x_i)} - 1). \quad (6)$$

Rani et al. [31] proposed the exponential entropy and distance measure for IVIFSs. Inspired by this, this study will discuss the problem of LIVqROFSs on entropy and distance measure. In the following, we give the axiomatic definition of entropy and distance measure for LIVqROFSs.

Definition 3.1. An entropy measure of $LIVqROFS(X)$ is a real-valued function $E : LIVqROFS(X) \to [0, 1]$, if it satisfies the following conditions:

- $(E1)$ $E(\hat{A}) = 0$ if and only if $\hat{A} = ([s_\tau, s_\tau], [s_0, s_0])$ or $\hat{A} = ([s_0, s_0], [s_\tau, s_\tau])$ for each $x_i \in X$(i.e. $\hat{A}$ is a crisp set);
- $(E2)$ $E(\hat{A}) = 1$ if and only if $[s_{\hat{\theta}_L}(x_i), s_{\hat{\theta}_U}(x_i)] = [s_{\hat{\sigma}_L}(x_i), s_{\hat{\sigma}_U}(x_i)]$ for each $x_i \in X$;
- $(E3)$ $E(\hat{A}) = E(\hat{A}^C)$;
- $(E4)$ $E(\hat{A}_1) \leq E(\hat{A}_2)$ if $\hat{A}_1 \leq \hat{A}_2$ when $s_{\hat{\theta}_{2L}}(x_i) \leq s_{\hat{\sigma}_{2L}}(x_i)$ and $s_{\hat{\theta}_{2U}}(x_i) \leq s_{\hat{\sigma}_{2U}}(x_i)$ for each $x_i \in X$, or $\hat{A}_2 \leq \hat{A}_1$ when $s_{\hat{\theta}_{2L}}(x_i) \geq s_{\hat{\sigma}_{2L}}(x_i)$ and $s_{\hat{\theta}_{2U}}(x_i) \geq s_{\hat{\sigma}_{2U}}(x_i)$.

Definition 3.2. Let $\hat{A}_1, \hat{A}_2, \hat{A}_3 \in LIVqROFS(X)$, a mapping $D : LIVqROFS(X) \times LIVqROFS(X) \to [0, 1]$ is called a distance measure on $LIVqROFS(X)$, if it satisfies the following properties:

- $(D1)$ $0 \leq D(\hat{A}_1, \hat{A}_2) \leq 1$;
- $(D2)$ $D(\hat{A}_1, \hat{A}_2) = 0$ if $\hat{A}_1 = \hat{A}_2$;
- $(D3)$ $D(\hat{A}_1, \hat{A}_2) = D(\hat{A}_2, \hat{A}_1)$;
- $(D4)$ If $\hat{A}_1 \leq \hat{A}_2 \leq \hat{A}_3$, then $D(\hat{A}_1, \hat{A}_2) \leq D(\hat{A}_1, \hat{A}_3)$ and $D(\hat{A}_2, \hat{A}_3) \leq D(\hat{A}_1, \hat{A}_3)$.

Definition 3.3. Let $s_\alpha \in \bar{S}$, $\phi : \bar{S} \to [0, \tau]$ is a mapping, such that $\phi(s_\alpha) = \alpha$.

Definition 3.4. Let $X = \{x_1, x_2, \cdots, x_n\}$, $\hat{A} \in LIVqROFS(X)$, the entropy measure is defined by

$$E(\hat{A}) = \frac{1}{\sqrt[q]{n}(\sqrt{e}-1)} \left\{ \sum_{i=1}^{n} \left(\Psi_{\hat{A}}(x_i)e^{(1-\Psi_{\hat{A}}(x_i))} + (1-\Psi_{\hat{A}}(x_i))e^{\Psi_{\hat{A}}(x_i)} - 1 \right)^q \right\}^{\frac{1}{q}}, \quad (7)$$

where

$$\Psi_{\hat{A}}(x_i) = \frac{\phi(s_{\hat{\theta}_L}(x_i)) + \phi(s_{\hat{\theta}_U}(x_i)) + 2\tau - \phi(s_{\hat{\sigma}_L}(x_i)) - \phi(s_{\hat{\sigma}_U}(x_i))}{4\tau}.$$

In the following, we prove the validity of our proposed entropy measure (7).

Theorem 3.5. *The mapping $E(\hat{A})$ defined in Equation (7) is a linguistic interval-valued q-Rung orthopair fuzzy entropy measure.*

Proof. Obviously to prove that $E(\hat{A})$ is an entropy measure of LIVqROFS(X), we just need to prove that it satisfies (E1)-(E4) in Definition 3.

(E1)Let $\hat{A}$ be a crisp set then we have $\hat{A} = ([s_\tau, s_\tau], [s_0, s_0])$ or $\hat{A} = ([s_0, s_0], [s_\tau, s_\tau])$, that is $[s_{\hat{\theta}_L}(x_i), s_{\hat{\theta}_U}(x_i)] = [s_\tau, s_\tau]$ and $[s_{\hat{\sigma}_L}(x_i), s_{\hat{\sigma}_U}(x_i)] = [s_0, s_0]$ or $[s_{\hat{\theta}_L}(x_i), s_{\hat{\theta}_U}(x_i)] = [s_0, s_0]$ and $[s_{\hat{\sigma}_L}(x_i), s_{\hat{\sigma}_U}(x_i)] = [s_\tau, s_\tau]$ for each $x_i \in X$, so we get $\Psi_{\hat{A}}(x_i) = 1$ or $\Psi_{\hat{A}}(x_i) = 0$. From equation (7), we obtain that $E(\hat{A}) = 0$.

On the other hand, if $E(\hat{A}) = 0$, from the definition of exponential entropy given by Pal in [41], we know that $\Psi_{\hat{A}}(x_i)=0$ or 1 for each $x_i \in X$. i.e.,

$$\frac{\phi(s_{\hat{\theta}_L}(x_i))+\phi(s_{\hat{\theta}_U}(x_i))+2\tau-\phi(s_{\hat{\sigma}_L}(x_i))-\phi(s_{\hat{\sigma}_U}(x_i))}{4\tau}=0 \quad (8)$$

or

$$\frac{\phi(s_{\hat{\theta}_L}(x_i))+\phi(s_{\hat{\theta}_U}(x_i))+2\tau-\phi(s_{\hat{\sigma}_L}(x_i))-\phi(s_{\hat{\sigma}_U}(x_i))}{4\tau}=1 \quad (9)$$

for each $x_i \in X$. Now Equations (8) and (9) hold if either $\hat{A}=([s_\tau, s_\tau],[s_0, s_0])$ or $\hat{A}=([s_0, s_0],[s_\tau, s_\tau])$ i.e. $\hat{A}$ is a crisp set.

(E2)Let $[s_{\hat{\theta}_L}(x_i), s_{\hat{\theta}_U}(x_i)]=[s_{\hat{\sigma}_L}(x_i), s_{\hat{\sigma}_U}(x_i)]$, i.e. $s_{\hat{\theta}_L}(x_i)=s_{\hat{\sigma}_L}(x_i)$, $s_{\hat{\theta}_U}(x_i)=s_{\hat{\sigma}_U}(x_i)$ for each $x_i \in X$. Apply the condition to Equation (7), we can get $E(\hat{A})=1$.

On the other hand, let

$$f(\Psi_{\hat{A}}(x_i)) =\frac{1}{(\sqrt{e}-1)}(\Psi_{\hat{A}}(x_i)e^{(1-\Psi_{\hat{A}}(x_i))} + (1-\Psi_{\hat{A}}(x_i))e^{\Psi_{\hat{A}}(x_i)}-1), \quad (10)$$

if $E(\hat{A})=1$, that is $\frac{1}{n}\sum_{i=1}^{n} f(\Psi_{\hat{A}}(x_i))=1$, so we can get $f(\Psi_{\hat{A}}(x_i))=1$ for each $x_i \in X$. Take the partial derivative of equation (10) with respect to $\Psi_{\hat{A}}(x_i)$ and set it equal to zero, we have

$$\frac{\partial f(\Psi_{\hat{A}}(x_i))}{\partial(\Psi_{\hat{A}}(x_i)))}=\frac{1}{(\sqrt{e}-1)}((1-\Psi_{\hat{A}}(x_i))e^{1-\Psi_{\hat{A}}(x_i)} - \Psi_{\hat{A}}(x_i)e^{\Psi_{\hat{A}}(x_i)})=0,$$

it implies $(1-\Psi_{\hat{A}}(x_i))e^{1-\Psi_{\hat{A}}(x_i)}=\Psi_{\hat{A}}(x_i)e^{\Psi_{\hat{A}}(x_i)}$ for each $x_i \in X$. Since $g(x)=xe^x$ is a bijection function, we get $1-\Psi_{\hat{A}}(x_i)=\Psi_{\hat{A}}(x_i)$, that is $\Psi_{\hat{A}}(x_i)=0.5$ for each $x_i \in X$. Since

$$\frac{\partial^2 f(\Psi_{\hat{A}}(x_i))}{\partial(\Psi_{\hat{A}}(x_i)))^2}=\frac{1}{(\sqrt{e}-1)}((\Psi_{\hat{A}}(x_i)-2)e^{(1-\Psi_{\hat{A}}(x_i))} - (1+\Psi_{\hat{A}}(x_i))e^{\Psi_{\hat{A}}(x_i)}),$$

we get

$$\left[\frac{\partial^2 f(\Psi_{\hat{A}}(x_i))}{\partial(\Psi_{\hat{A}}(x_i)))^2}\right]_{\Psi_{\hat{A}}(x_i)=0.5}<0,$$

for each $x_i \in X$. So $f(\Psi_{\hat{A}}(x_i))$ is a concave function and has a maximum at $\Psi_{\hat{A}}(x_i)=0.5$. By Equation (7), $E(\hat{A})$ achieves the maximum at $\Psi_{\hat{A}}(x_i)=0.5$ which implies that $[s_{\hat{\theta}_L}(x_i), s_{\hat{\theta}_U}(x_i)]=[s_{\hat{\sigma}_L}(x_i), s_{\hat{\sigma}_U}(x_i)]$.

(E3) Form $\hat{A}^C=\{(x_i,[s_{\hat{\sigma}_L}(x_i), s_{\hat{\sigma}_U}(x_i)], [s_{\hat{\theta}_L}(x_i), s_{\hat{\theta}_U}(x_i)])|x_i \in X\}$ and Equation (7), we can easily get that $E(\hat{A})=E(\hat{A}^C)$.

(E4)If we take $x=\phi(s_{\hat{\theta}_L}(x_i))+\phi(s_{\hat{\theta}_U}(x_i))$ and $y=\phi(s_{\hat{\sigma}_L}(x_i))+\phi(s_{\hat{\sigma}_U}(x_i))$, let

$$g(x,y)=(\frac{x+2\tau-y}{4\tau})e^{(1-\frac{x+2\tau-y}{4\tau})} + (1-\frac{x+2\tau-y}{4\tau})e^{(\frac{x+2\tau-y}{4\tau})}-1, \quad (11)$$

where $x, y \in [0,\tau]$.

Taking the partial derivatives of g with respect to x and y, respectively, we get

$$\frac{\partial g}{\partial x}=\frac{1}{4\tau}\left[(\frac{y+2\tau-x}{4\tau})e^{(\frac{y+2\tau-x}{4\tau})} - \frac{x+2\tau-y}{4\tau}e^{(\frac{x+2\tau-y}{4\tau})}\right], \quad (12)$$

$$\frac{\partial g}{\partial y}=\frac{1}{4\tau}\left[(\frac{x+2\tau-y}{4\tau})e^{(\frac{x+2\tau-y}{4\tau})} - \frac{y+2\tau-x}{4\tau}e^{(\frac{y+2\tau-x}{4\tau})}\right], \quad (13)$$

Set $\frac{\partial g}{\partial x}=0$ and $\frac{\partial g}{\partial y}=0$ to find the critical points, we get $x=y$. From Equation (12) and $x=y$, we obtain

$$\frac{\partial g}{\partial x}\geq 0, \text{ when } x\leq y$$

and

$$\frac{\partial g}{\partial x}\leq 0, \text{ when } x\geq y,$$

for any $x, y \in [0,\tau]$.

Thus, $g(x,y)$ is increasing with respect to x when $x\leq y$ and is decreasing when $x\geq y$. Similarly, we obtain that

$$\frac{\partial g}{\partial y}\leq 0, \text{ when } x\leq y$$

and

$$\frac{\partial g}{\partial y}\geq 0, \text{ when } x\geq y.$$

Now if $\hat{A}_1\leq\hat{A}_2$, with $s_{\hat{\theta}_{2L}}(x_i)\leq s_{\hat{\sigma}_{2L}}(x_i)$ and $s_{\hat{\theta}_{2U}}(x_i)\leq s_{\hat{\sigma}_{2U}}(x_i)$ for each $x_i \in X$. Then we have

$$s_{\hat{\theta}_{1L}(x_i)}\leq s_{\hat{\theta}_{2L}(x_i)}\leq s_{\hat{\sigma}_{2L}(x_i)}\leq s_{\hat{\sigma}_{1L}(x_i)},$$

$$s_{\hat{\theta}_{1U}(x_i)}\leq s_{\hat{\theta}_{2U}(x_i)}\leq s_{\hat{\sigma}_{2U}(x_i)}\leq s_{\hat{\sigma}_{1U}(x_i)}.$$

It implies that $s_{\hat\theta_{1L}(x_i)} \le s_{\hat\sigma_{1L}(x_i)}$, and $s_{\hat\theta_{1U}(x_i)} \le s_{\hat\sigma_{1U}(x_i)}$. Thus, from the monotonic of $g(x, y)$ and Equation (7), we obtain $E(\hat A_1) \le E(\hat A_2)$.

Similarly, when $\hat A_2 \le \hat A_1$ with $s_{\hat\theta_{2L}}(x_i) \ge s_{\hat\sigma_{2L}}(x_i)$ and $s_{\hat\theta_{2U}}(x_i) \ge s_{\hat\sigma_{2U}}(x_i)$ for each $x_i \in X$ and thus, it can be prove that $E(\hat A_1) \le E(\hat A_2)$.

By calculating the weight of each element $x_i \in X$, a weighted exponential entropy measure of LIVqROFS is proposed as follows:

$$E_W(\hat A) = \frac{1}{\sqrt{e}-1}\left\{\sum_{i=1}^{n} w_i\left(\Psi_{\hat A}(x_i)e^{(1-\Psi_{\hat A}(x_i))} + (1-\Psi_{\hat A}(x_i))e^{\Psi_{\hat A}(x_i)} - 1\right)^q\right\}^{\frac{1}{q}}, \quad (14)$$

where $w_i \ge 0$, $\sum_{i=1}^{n} w_i = 1$, and

$$\Psi_{\hat A}(x_i) = \frac{\phi(s_{\hat\theta_L}(x_i)) + \phi(s_{\hat\theta_U}(x_i)) + 2\tau - \phi(s_{\hat\sigma_L}(x_i)) - \phi(s_{\hat\sigma_U}(x_i))}{4\tau}.$$

It is clear, if $w_i = \frac{1}{n}$, then $E_W(\hat A) = E(\hat A)$. It can be easily checked that the mapping $E_W(\hat A)$, defined by Equation (14), is an entropy measure for LIVqROFS.

For $A_1, A_2 \in FS(X)$, $X = \{x_1, x_2, \cdots, x_n\}$, Fan and Xie [42] proposed the fuzzy information for discrimination of $\hat A_1$ against $\hat A_2$ is defined by

$$I_1(A_1, A_2) = \sum_{i=1}^{n}(1 - (1-\mu_{A_1}(x_i))e^{(\mu_{A_1}(x_i)-\mu_{A_2}(x_i))} - \mu_{A_1}(x_i)e^{(\mu_{A_2}(x_i)-\mu_{A_1}(x_i))}). \quad (15)$$

The fuzzy distance between A_1 and A_2 is defined by

$$D_1(A_1, A_2) = I_1(A_1, A_2) + I_1(A_2, A_1). \quad (16)$$

For $\hat A_1, \hat A_2 \in IVIFS(X)$, $X = \{x_1, x_2, \cdots, x_n\}$, Ohlan [34] proposed the interval-valued intuitionistic fuzzy information for discrimination of $\hat A_1$ against $\hat A_2$ is defined by

$$I_2(\hat A_1, \hat A_2) = \sum_{i=1}^{n}(1 - (1 - H_{\hat A_1}(x_i)) e^{(H_{\hat A_1}(x_i)-H_{\hat A_2}(x_i))} - H_{\hat A_1}(x_i)e^{(H_{\hat A_2}(x_i)-H_{\hat A_1}(x_i))}), \quad (17)$$

where

$$H_{\hat A_1}(x_i) = \frac{\hat\theta_{1L}(x_i) + \hat\theta_{1U}(x_i) + 2 - \hat\sigma_{1L}(x_i) - \hat\sigma_{1U}(x_i)}{4},$$

$$H_{\hat A_2}(x_i) = \frac{\hat\theta_{2L}(x_i) + \hat\theta_{2U}(x_i) + 2 - \hat\sigma_{2L}(x_i) - \hat\sigma_{2U}(x_i)}{4}.$$

The distance measure for IVIFSs is defined as

$$D_2(\hat A_1, \hat A_2) = I_2(\hat A_1, \hat A_2) + I_2(\hat A_2, \hat A_1). \quad (18)$$

Inspired by the above distance measures of FSs and IVIFSs, now the distance measures of LIVqROFSs can be defined as the following.

Definition 3.6. Let $X = \{x_1, x_2, \cdots, x_n\}$, $\hat A_1, \hat A_2 \in LIVqROFS(X)$, the linguistic interval-valued q-Rung orthopair fuzzy information for discrimination of $\hat A_1$ against $\hat A_2$ is defined by

$$I(\hat A_1, \hat A_2) = \left\{\sum_{i=1}^{n}\left(1 - (1-\Psi_{\hat A_1}(x_i)) e^{(\Psi_{\hat A_1}(x_i)-\Psi_{\hat A_2}(x_i))} - \Psi_{\hat A_1}(x_i)e^{(\Psi_{\hat A_2}(x_i)-\Psi_{\hat A_1}(x_i))}\right)^q\right\}^{\frac{1}{q}}, \quad (19)$$

where

$$\Psi_{\hat A_1}(x_i) = \frac{\phi(s_{\hat\theta_{1L}}(x_i)) + \phi(s_{\hat\theta_{1U}}(x_i)) + 2\tau - \phi(s_{\hat\sigma_{1L}}(x_i)) - \phi(s_{\hat\sigma_{1U}}(x_i))}{4\tau},$$

$$\Psi_{\hat A_2}(x_i) = \frac{\phi(s_{\hat\theta_{2L}}(x_i)) + \phi(s_{\hat\theta_{2U}}(x_i)) + 2\tau - \phi(s_{\hat\sigma_{2L}}(x_i)) - \phi(s_{\hat\sigma_{2U}}(x_i))}{4\tau}.$$

Theorem 3.7. *If $q = 1$, the relation between $E(\hat A)$ and $I(\hat A, \hat B)$ is given by the following formula*

$$E(\hat A) = 1 - \frac{\sqrt{e}}{n(\sqrt{e}-1)} I(\hat A, \hat B),$$

where $\Psi_{\hat B}(x_i) = \frac{1}{2}$.

Proof.

$$\begin{aligned}
&\sqrt{e}I(\hat{A},\hat{B})\\
&=\sqrt{e}\sum_{i=1}^{n}(1-(1-\Psi_{\hat{A}}(x_i))e^{(\Psi_{\hat{A}}(x_i)-\Psi_{\hat{B}}(x_i))}-\Psi_{\hat{A}}(x_i)e^{(\Psi_{\hat{B}}(x_i)-\Psi_{\hat{A}}(x_i))})\\
&=\sqrt{e}\sum_{i=1}^{n}(1-(1-\Psi_{\hat{A}}(x_i))e^{(\Psi_{\hat{A}}(x_i)-\frac{1}{2})}-\Psi_{\hat{A}}(x_i)e^{(\frac{1}{2}-\Psi_{\hat{A}}(x_i))})\\
&=\sum_{i=1}^{n}(\sqrt{e}-(1-\Psi_{\hat{A}}(x_i))e^{\Psi_{\hat{A}}(x_i)}-\Psi_{\hat{A}}(x_i)e^{(1-\Psi_{\hat{A}}(x_i))})\\
&=n(\sqrt{e}-1)-\sum_{i=1}^{n}((1-\Psi_{\hat{A}}(x_i))e^{\Psi_{\hat{A}}(x_i)}+\Psi_{\hat{A}}(x_i)e^{(1-\Psi_{\hat{A}}(x_i))}-1)\\
&=n(\sqrt{e}-1)-\sum_{i=1}^{n}((1-\Psi_{\hat{A}}(x_i))e^{\Psi_{\hat{A}}(x_i)}+\Psi_{\hat{A}}(x_i)e^{(1-\Psi_{\hat{A}}(x_i))}-1)\\
&=-\sum_{i=1}^{n}((1-\Psi_{\hat{A}}(x_i))e^{\Psi_{\hat{A}}(x_i)}+\Psi_{\hat{A}}(x_i)e^{(1-\Psi_{\hat{A}}(x_i))}-1)\\
&=n(\sqrt{e}-1)-n(\sqrt{e}-1)E(\hat{A})\\
&=n(\sqrt{e}-1)(1-E(\hat{A})).
\end{aligned}$$

Thus, we get $E(\hat{A})=1-\frac{\sqrt{e}}{n(\sqrt{e}-1)}I(\hat{A},\hat{B})$.

Definition 3.8. Let $X=\{x_1,x_2,\cdots,x_n\}$, $\hat{A}_i=(s_{\hat{\theta}_i},s_{\hat{\sigma}_i})$, $i=1,2$ be two LIVqROFSs, the distance measure $D(\hat{A}_1,\hat{A}_2)$ between the $LIVqROFS(X)$ A_1 and A_2 is defined as follows:

$$\begin{aligned}
D(\hat{A}_1,\hat{A}_2)&=\frac{1}{2\sqrt[q]{n}(1-e^{-1})}\\
&\left\{\sum_{i=1}^{n}\left(2-(1-\Psi_{\hat{A}_1}(x_i)+\Psi_{\hat{A}_2}(x_i))e^{(\Psi_{\hat{A}_1}(x_i)-\Psi_{\hat{A}_2}(x_i))}\right.\right.\\
&\left.\left.-(1-\Psi_{\hat{A}_2}(x_i)+\Psi_{\hat{A}_1}(x_i))e^{(\Psi_{\hat{A}_2}(x_i)-\Psi_{\hat{A}_1}(x_i))}\right)^{q}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{20}$$

In particular, if $q=1$, then

$$D(\hat{A}_1,\hat{A}_2)=I(\hat{A}_1,\hat{A}_2)+I(\hat{A}_2,\hat{A}_1). \tag{21}$$

Theorem 3.9. *The mapping $D(\hat{A}_1,\hat{A}_2)$ defined in Equation (20) is a distance measure on $LIVqROFS(X)$.*

Proof. In order for Equation (20) to be qualified as a sensible measure of $LIVqROFSs$, it must satisfy the conditions (D1)-(D4) in Definition 3.2.

(D1)It is clear that the function $h(t)=2-(1-t)e^{t}-(1+t)e^{-t}$ in interval [-1,1] has maximum value $f_{max}(t)=2-2e^{-1}$ and minimum value $f_{min}(t)=0$. $h(t)$ is decreasing in [-1,0] and is increasing in [0,1]. Since $-1\le\Psi_{\hat{A}_1}(x_i)-\Psi_{\hat{A}_2}(x_i)\le 1$, we get $0\le D(\hat{A}_1,\hat{A}_2)\le 1$.

(D2)Let $\hat{A}_1$ and $\hat{A}_2$ be two $LIVqROFSs$, if $\hat{A}_1=\hat{A}_2$ then $\phi(s_{\hat{\theta}_{1L}}(x_i))=\phi(s_{\hat{\theta}_{2L}}(x_i))$, $\phi(s_{\hat{\theta}_{1U}}(x_i))=\phi(s_{\hat{\theta}_{2U}}(x_i))$,$\phi(s_{\hat{\sigma}_{1L}}(x_i))=\phi(s_{\hat{\sigma}_{2L}}(x_i))$, $\phi(s_{\hat{\sigma}_{1U}}(x_i))=\phi(s_{\hat{\sigma}_{2U}}(x_i))$. It is clear $\Psi_{\hat{A}_1}(x_i)-\Psi_{\hat{A}_2}(x_i)=0$, therefore, $D(\hat{A}_1,\hat{A}_2)=0$.

(D3)$D(\hat{A}_1,\hat{A}_2)=D(\hat{A}_2,\hat{A}_1)$ is easily known from Equation (20).

(D4)If $\hat{A}_1\le\hat{A}_2\le\hat{A}_3$, we have $0\le\Psi_{\hat{A}_2}(x_i)-\Psi_{\hat{A}_1}(x_i)\le\Psi_{\hat{A}_3}(x_i)-\Psi_{\hat{A}_1}(x_i)$ and $0\le\Psi_{\hat{A}_3}(x_i)-\Psi_{\hat{A}_2}(x_i)\le\Psi_{\hat{A}_3}(x_i)-\Psi_{\hat{A}_1}(x_i)$. Then it is easy to see that $D(\hat{A}_1,\hat{A}_2)\le D(\hat{A}_1,\hat{A}_3)$ and $D(\hat{A}_2,\hat{A}_3)\le D(\hat{A}_1,\hat{A}_3)$.

It can be said that $D(\hat{A}_1,\hat{A}_2)$ is a distance measure between LIVqROFSs $\hat{A}_1$ and $\hat{A}_2$ since $D(\hat{A}_1,\hat{A}_2)$ satisfies (D1)-(D4). □

4. Application of the proposed entropy measure of LIVqROFSs

Now we apply the proposed LIVqROFSs based weighted entropy measure to the Multi-criteria group decision making (MCGDM) problem. Therefore, we introduce a MCGDM model based on LIVqROFS weighted entropy measure.

4.1. A method of group decision making of LIVqROFS setting based on the proposed weighted entropy measure

We provide a group decision making method based on LIVqROFSs with the known experts and unknown criteria weights. Let $\hat{\mathcal{X}} = \{\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_m\}$ be a set of m feasible alternatives, and $\hat{\mathcal{C}} = \{\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_n\}$ be a set of attributes. $w = (w_1, w_2, \cdots, w_n)^T$ is the weighting vector of attributes satisfying $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Let $D = \{d_1, d_2, \cdots, d_l\}$ be a set of decision makers with the weighting vector $V = (v_1, v_2, \cdots, v_l)^T$ satisfying $v_k \geq 0$ and $\sum_{k=1}^{l} v_k = 1$. Assume that each decision maker gives their own decision matrix $R_k = \left(\hat{r}_{ij}^{(k)}\right)_{m\times n}$, where

$$\hat{r}_{ij}^{(k)} = ([s_{\hat{\theta}_{ijL}^{(k)}}, s_{\hat{\theta}_{ijU}^{(k)}}], [s_{\hat{\sigma}_{ijL}^{(k)}}, s_{\hat{\sigma}_{ijU}^{(k)}}])$$

is the evaluation result given by decision maker $d_k \in D$ under the attribute $\hat{c}_j \in \hat{\mathcal{C}}$ for the alternative $\hat{x}_i \in \hat{\mathcal{X}}$.

Step 1. Construct the linguistic interval-valued q-rung orthopair fuzzy decision matrices $R_k = \left(\hat{r}_{ij}^{(k)}\right)_{m\times n}$ $(k = 1, \cdots, l)$.

Step 2. Based on the given expert weights, apply the proposed LIVqROF operators(Khan et al. [13]) to aggregate the LIVqROFVs. We choose the LIVqROFWA operator to aggregate all the decision matrices R_k into a collective decision matrix $R = (\hat{r}_{ij})_{m\times n}$, where

$$\begin{aligned}\hat{r}_{ij} &= ([s_{\hat{\theta}_{ijL}}, s_{\hat{\theta}_{ijU}}], [s_{\hat{\sigma}_{ijL}}, s_{\hat{\sigma}_{ijU}}])\\ &= LIVqROFWA(\hat{r}_{ij}^{(1)}, \hat{r}_{ij}^{(2)}, \cdots, \hat{r}_{ij}^{(l)})\\ &= v_1\hat{r}_{ij}^{(1)} \oplus v_2\hat{r}_{ij}^{(2)} \oplus \cdots \oplus v_l\hat{r}_{ij}^{(l)}\\ &= ([s_{\tau\sqrt[q]{1-\prod_{k=1}^{l}(1-(\frac{\hat{\theta}_{ijL}^{(k)}}{\tau})^q)^{v_k}}}, s_{\tau\sqrt[q]{1-\prod_{k=1}^{l}(1-(\frac{\hat{\theta}_{ijU}^{(k)}}{\tau})^q)^{v_k}}}],\\ &\quad [s_{\tau\prod_{k=1}^{l}(\frac{\hat{\sigma}_{ijL}^{(k)}}{\tau})^{v_k}}, s_{\tau\prod_{k=1}^{l}(\frac{\hat{\sigma}_{ijU}^{(k)}}{\tau})^{v_k}}]).\end{aligned}$$

Step 3. Calculate the weight vector of the criterion. The weight formula of the criterion is determined as follows.

$$w_j = \frac{1 - E(\hat{c}_j)}{n - \sum_{j=1}^{n} E(\hat{c}_j)}, j = 1, 2, \cdots, n; \tag{22}$$

where $E(\hat{c}_j)$ is the entropy value of the jth attribute, which is calculated by Equation (7).

Step 4. For each alternative $\hat{x}_i$, Equation (14) is used to calculate the weighted LIVqROF information measure.

Step 5. Rank all the alternative $\hat{x}_i$ according to the $E_W(\hat{x}_i)$, the smaller the value of $E_W(\hat{x}_i)$, the better the alternative $\hat{x}_i$.

4.2. Illustrative example

In the following, we apply the proposed group decision method to the evaluation of college teachers. Five college teachers $\{\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5\}$ who participated in the evaluation from four main evaluation indexes: Teaching ability $(\hat{c}_1)$, student training and service $(\hat{c}_2)$,cientific research ability $(\hat{c}_3)$, discipline construction $(\hat{c}_4)$. Assume that, three decision makers $\{d_1, d_2, d_3\}$ evaluated five teachers with four attributes using the following linguistic scale

$$\begin{aligned}S = \{&s_0 = \textit{extrmely poor}, s_1 = \textit{very poor},\\ &s_2 = \textit{poor}, s_3 = \textit{slightly poor},\\ &s_4 = \textit{fair}, s_5 = \textit{slightly good}, s_6 = \textit{good},\\ &s_7 = \textit{very good}, s_8 = \textit{extremly good}\}.\end{aligned}$$

Step 1. The decision matrices given by decision makers are shown in Tables 1, 2 and 3 .

Step 2. The weighting vector $v = (0.3, 0.5, 0.2)$ of decision maker is given. Let $q = 3$, the LIVqROFWA operator is utilized to aggregate all the decision matrices $R_k(k = 1, 2, 3)$ into a collective decision matrix R, as shown in Table 4.

Step 3. By using Equation (22), take $q = 3$, we get the criteria weight as:

$$w = (0.2729, 0.1427, 0.3908, 0.1936).$$

Table 1
Decision matrix R_1 provided by decision maker d_1

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$	$\hat{c}_4$
$\hat{x}_1$	$([s_2, s_3], [s_3, s_4])$	$([s_3, s_4], [s_1, s_4])$	$([s_2, s_6], [s_1, s_3])$	$([s_3, s_5], [s_2, s_4])$
$\hat{x}_2$	$([s_2, s_4], [s_2, s_6])$	$([s_2, s_6], [s_1, s_3])$	$([s_3, s_5], [s_2, s_5])$	$([s_3, s_6], [s_3, s_5])$
$\hat{x}_3$	$([s_3, s_5], [s_2, s_6])$	$([s_4, s_6], [s_3, s_5])$	$([s_3, s_4], [s_1, s_2])$	$([s_3, s_5], [s_2, s_6])$
$\hat{x}_4$	$([s_1, s_3], [s_3, s_4])$	$([s_3, s_5], [s_2, s_4])$	$([s_1, s_3], [s_3, s_6])$	$([s_3, s_4], [s_2, s_6])$
$\hat{x}_5$	$([s_2, s_5], [s_3, s_6])$	$([s_2, s_4], [s_2, s_6])$	$([s_2, s_6], [s_3, s_5])$	$([s_2, s_4], [s_3, s_6])$

Table 2
Decision matrix R_2 provided by decision maker d_2

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$	$\hat{c}_4$
$\hat{x}_1$	$([s_2, s_4], [s_3, s_5])$	$([s_2, s_4], [s_2, s_6])$	$([s_2, s_3], [s_3, s_4])$	$([s_3, s_5], [s_1, s_4])$
$\hat{x}_2$	$([s_3, s_6], [s_2, s_5])$	$([s_3, s_5], [s_3, s_5])$	$([s_3, s_6], [s_3, s_5])$	$([s_4, s_6], [s_3, s_4])$
$\hat{x}_3$	$([s_3, s_5], [s_2, s_6])$	$([s_2, s_5], [s_3, s_4])$	$([s_3, s_4], [s_1, s_3])$	$([s_3, s_6], [s_2, s_5])$
$\hat{x}_4$	$([s_2, s_3], [s_3, s_5])$	$([s_3, s_5], [s_4, s_5])$	$([s_1, s_4], [s_2, s_6])$	$([s_2, s_6], [s_2, s_5])$
$\hat{x}_5$	$([s_3, s_4], [s_2, s_5])$	$([s_3, s_4], [s_3, s_6])$	$([s_3, s_6], [s_3, s_5])$	$([s_2, s_5], [s_4, s_5])$

Table 3
Decision matrix R_3 provided by decision maker d_3

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$	$\hat{c}_4$
$\hat{x}_1$	$([s_1, s_4], [s_3, s_5])$	$([s_2, s_4], [s_3, s_6])$	$([s_2, s_3], [s_4, s_5])$	$([s_3, s_4], [s_1, s_5])$
$\hat{x}_2$	$([s_3, s_6], [s_2, s_5])$	$([s_2, s_6], [s_3, s_5])$	$([s_2, s_5], [s_3, s_6])$	$([s_2, s_4], [s_3, s_5])$
$\hat{x}_3$	$([s_3, s_6], [s_1, s_4])$	$([s_3, s_5], [s_2, s_6])$	$([s_3, s_5], [s_2, s_3])$	$([s_2, s_6], [s_4, s_5])$
$\hat{x}_4$	$([s_2, s_3], [s_3, s_5])$	$([s_3, s_4], [s_3, s_5])$	$([s_4, s_6], [s_4, s_5])$	$([s_1, s_5], [s_3, s_4])$
$\hat{x}_5$	$([s_2, s_4], [s_3, s_6])$	$([s_1, s_5], [s_3, s_6])$	$([s_3, s_4], [s_3, s_5])$	$([s_2, s_5], [s_1, s_3])$

Table 4
Collective LIVqROF assessment information

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$	$\hat{c}_4$
$\hat{x}_1$	$([s_{1.8765}, s_{3.7612}], [s_{3.0000}, s_{4.6762}])$	$([s_{2.3973}, s_{4.0000}], [s_{1.7617}, s_{5.3128}])$	$([s_{2.0000}, s_{4.5433}], [s_{2.2855}, s_{3.8367}])$	$([s_{3.0000}, s_{4.8418}], [s_{1.2311}, s_{4.1826}])$
$\hat{x}_2$	$([s_{2.7753}, s_{5.6127}], [s_{2.0000}, s_{5.2811}])$	$([s_{2.6008}, s_{5.5779}], [s_{2.1577}, s_{4.2896}])$	$([s_{2.8545}, s_{5.5779}], [s_{2.6564}, s_{5.1857}])$	$([s_{3.4835}, s_{5.7531}], [s_{3.0000}, s_{4.4721}])$
$\hat{x}_3$	$([s_{3.0000}, s_{5.2560}], [s_{1.7411}, s_{5.5326}])$	$([s_{3.0802}, s_{5.3706}], [s_{2.7663}, s_{4.6382}])$	$([s_{3.0000}, s_{4.2529}], [s_{1.1487}, s_{2.6564}])$	$([s_{2.8545}, s_{5.7611}], [s_{2.2974}, s_{5.2811}])$
$\hat{x}_4$	$([s_{1.8080}, s_{3.0000}], [s_{3.0000}, s_{4.6762}])$	$([s_{3.0000}, s_{4.8418}], [s_{3.0673}, s_{4.6762}])$	$([s_{2.4256}, s_{4.4764}], [s_{2.5946}, s_{5.7852}])$	$([s_{2.3146}, s_{5.4101}], [s_{2.1689}, s_{5.0506}])$
$\hat{x}_5$	$([s_{2.6008}, s_{4.3664}], [s_{2.4495}, s_{5.4772}])$	$([s_{2.5316}, s_{4.2529}], [s_{2.6564}, s_{6.000}])$	$([s_{4.7083}, s_{6.0774}], [s_{3.0000}, s_{5.0000}])$	$([s_{2.0000}, s_{4.7567}], [s_{2.7808}, s_{4.7682}])$

Step 4. Take $q = 3$, calculate $E_w(\hat{x}_i)$ for each the alternative using Equation (14).

$$E_w(\hat{x}_1) = 0.9911,\ E_w(\hat{x}_2) = 0.9944,\ E_w(\hat{x}_3) = 0.9807,\ E_w(\hat{x}_4) = 0.9884,\ E_w(\hat{x}_5) = 0.9859.$$

Step 5. Since $E_w(\hat{x}_3) < E_w(\hat{x}_5) < E_w(\hat{x}_4) < E_w(\hat{x}_1) < E_w(\hat{x}_2)$. We get a descending order of $\hat{x}_i$.

$$\hat{x}_3 \succ \hat{x}_5 \succ \hat{x}_4 \succ \hat{x}_1 \succ \hat{x}_2.$$

Therefore, we get $\hat{x}_3$ is the best alternative.

4.3. Comparative analyses

In this subsection, we compare the proposed weighted entropy measure model for LIVqROFSs with other decision tools VIKOR model for LIVqROFSs proposed by Khan et al. in [13] and TOPSIS model for LIVqROFSs proposed by Gurmani et al. in [38] to illustrate the effectiveness of our proposed method.

Steps of Khan's approach are as under:

Step 1. Construct the decision matrices. For better comparison, we select the data in Tables 1, 2 and 3.

Step 2. The collective matrix is presented in Table 4 using the given expert weights.

Step 3. According to Table 4, we get the LIVqROFS positive ideal solution and LIVqROFS negative ideal solution respectively as:

$$r^{+} = \{([s_{3.0000}, s_{5.6127}], [s_{1.7411}, s_{4.6762}]),$$
$$([s_{3.0802}, s_{5.5779}], [s_{1.7617}, s_{4.2896}]),$$
$$([s_{4.7083}, s_{6.0774}], [s_{1.1487}, s_{2.6564}]),$$
$$([s_{3.4835}, s_{5.7611}], [s_{1.2311}, s_{4.1826}])\}$$

$$r^{-} = \{([s_{1.8080}, s_{3.0000}], [s_{3.0000}, s_{5.5326}]),$$
$$([s_{2.3973}, s_{4.0000}], [s_{3.0673}, s_{6.0000}]),$$
$$([s_{2.0000}, s_{4.2529}], [s_{3.0000}, s_{5.7852}]),$$
$$([s_{2.0000}, s_{4.7567}], [s_{3.0000}, s_{5.2811}])\}$$

Step 4. For better comparison, we give the criteria weight as:

$$w = (0.2729, 0.1427, 0.3908, 0.1936).$$

Step 5. Calculate the distance between the LIVqROFS positive ideal solution r^{+} and the alternative $\hat{x}_i$, and the distance between the LIVqROFS negative ideal solution r^{-} and the alternative $\hat{x}_i$ respectively as:

$$d_1^{+}(\hat{x}_1, r^{+}) = 0.1570, d_1^{-}(\hat{x}_1, r^{-}) = 0.1235;$$

$$d_2^{+}(\hat{x}_2, r^{+}) = 0.1010, d_2^{-}(\hat{x}_2, r^{-}) = 0.1065;$$

$$d_3^{+}(\hat{x}_3, r^{+}) = 0.1537, d_3^{-}(\hat{x}_3, r^{-}) = 0.1852;$$

$$d_4^{+}(\hat{x}_4, r^{+}) = 0.1437, d_4^{-}(\hat{x}_4, r^{-}) = 0.0650;$$

$$d_5^{+}(\hat{x}_5, r^{+}) = 0.1072, d_5^{-}(\hat{x}_5, r^{-}) = 0.1172.$$

Step 6. Calculate the relative closeness coefficient of each alternatives as follows:

$$\Lambda(\hat{x}_1) = 0.4403, \Lambda(\hat{x}_2) = 0.5133, \Lambda(\hat{x}_3) = 0.5465,$$
$$\Lambda(\hat{x}_4) = 0.3115, \Lambda(\hat{x}_5) = 0.5222.$$

Step 7. Rank the relative closeness coefficients in descending order to select the best alternative.

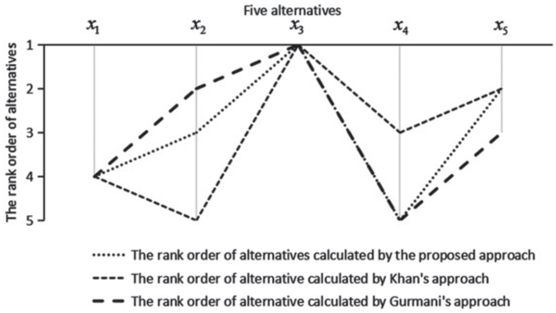

Fig. 1. The ranking of alternatives obtained by the three approaches.

$$\Lambda(\hat{x}_3) > \Lambda(\hat{x}_5) > \Lambda(\hat{x}_2) > \Lambda(\hat{x}_1) > \Lambda(\hat{x}_4).$$

Therefore, we get $\hat{x}_3$ is the best alternative.

Steps of Gurmani's approach are as under:

In order to make the comparison result more reasonable, we chose the same data, so steps 1, 2, 3 and 4 are the same as Khan's approach.

Step 5. Calculate the group utility measure Φ_i and individual regret measure y_i of alternatives $\hat{x}_i$ respectively as:

$$\Phi_1 = 0.5971, \Phi_2 = 0.3393, \Phi_3 = 0.4214,$$
$$\Phi_4 = 0.7355, \Phi_5 = 0.5832.$$

$$y_1 = 0.2421, y_2 = 0.2375, y_3 = 0.1978,$$
$$y_4 = 0.3657, y_5 = 0.1833.$$

Step 6. Calculate the compromise measure $\mathcal{Q}_i$ of alternatives $\hat{x}_i$.

$$\mathcal{Q}_1 = 0.4865, \mathcal{Q}_2 = 0.1485, \mathcal{Q}_3 = 0.1434,$$
$$\mathcal{Q}_4 = 1, \mathcal{Q}_5 = 0.3077.$$

Step 7. Rank the compromise measure in ascending order to select the best alternative.

$$\mathcal{Q}_3 < \mathcal{Q}_2 < \mathcal{Q}_5 < \mathcal{Q}_1 < \mathcal{Q}_4.$$

Therefore, we get $\hat{x}_3$ is the best alternative.

The comparison results of the proposed approach and the existing approachs are shown in Fig. 1.

According to the optimal results of the three methods, there are some similarities among the three methods, but there are also differences among the three methods from the final ranking of the alternatives. The group decision making method based on weighted exponential entropy measure in LIVqROFS environment is more efficient, simple and consistent than the existing measures and methods to solve the decision problem.

5. Conclusion

In this study, a new multi-attribute group decision making method is proposed under the linguistic interval-valued q-rung orthopair fuzzy environment. We propose novel entropy and distance measure for linguistic interval-valued q-rung orthopair fuzzy sets. The main innovations and advantages of this study are shown as follows:

(1) The axiomatic definitions of entropy and distance are proposed under the LIVqROF environment. It provides some important and reliable reference results for the subsequent research of complex information measures and distance measures in the LIVqROF environment.

(2) The proposed information measure is applied to the problem of multi-attribute decision making. An effective linguistic multi-attribute decision making model is established, which enriches the theory and method of qualitative decision making in the LIVqROF environment.

Theoretical analysis and numerical results show that the method is simple and intuitive without information loss. The model can be applied to medical diagnosis, personnel assessment, pollution treatment, quality evaluation and other fields. We will continue to extend the proposed method to complex decision information environment.

Conflicts of interest

I declare that I have no financial and personal relationships with other people or organizations that can inappropriately influence my work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled.

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