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Statistical properties and applications of the modified beta weibull family of distributions to engineering data

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Abstract. In this work, a new family of distribution, which generalizes the Beta Weibull-G family by the introduction of a shape parameter to enhance better fit and flexibility, called the Modified Beta Weibull-G family of distributions is obtained. The mixture representation of the derived family of distributions was discussed, with the results effective in studying moments, moment generating functions, order statistics. Parameters of the family of distributions were estimated using the maximum likelihood estimation method. By utilizing this modified class of distributions, we build a new distribution called the modified beta Weibull Weibull and applied it to engineering datasets. Application revealed a better performance in model fit, compared to some other distributions.

Keywords: Modified Beta-G Distribution, Weibull-G, Modified Beta Weibull G distribution, estimation, real life data

1. Introduction

Of recent, many distributions are developed by adding parameter(s) to the baseline distribution, which are more flexible and give A better fit to real-time data. These distributions have gained application to real life situations in engineering, health, finance, education, environmental sciences, economics, e.t.c. Examples of families of distribution include Beta-G ([14]), Weibull-G ([9]), Poisson-G ([1]), Exponentiated-G ([16]),Transmuted-G ([26]), Cubic Transmuted-G ([18]), Exponentiated Chen-G ([7]) to mention a few. With these generated families of distributions, researchers have been enabled to develop new distributions. The generated families have attracted many researchers due to the availability of computational and analytical facilities in most symbolic computation software platforms. Furthermore, with the aid of the mixture representation of the Exponentiated-G distributions, several properties of the derived distribution have been extensively studied.

The Weibull-H family of distributions has been extensively studied and applied to real life datasets due to its flexibility. Some of the studies had considered using the family of distributions in deriving extensions of some baseline distributions. Some of the extended distributions include Weibull-Gamma [5], Weibull-Burr ([6]), Weibull-sigma distribution ([20]), Weibull Exponential distribution ([30]) e.t.c. Given a baseline cumulative distribution function (c.d.f) G(x;σ) with p.d.f g(x;σ), the c.d.f of the Weibull-H distribution ([9]) is

$$H(x;\sigma) = 1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}} \tag{1}$$

and

$$h(x;\sigma) = \alpha\rho g(x;\sigma)\frac{G(x;\sigma)^{\rho-1}}{S(x;\sigma)^{\rho+1}}e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}} \tag{2}$$

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where

$$S(x;\sigma) = 1 - G(x;\sigma)$$

and x>0, α,ϕ >0. In a bid to obtain a family of distributions that will further extend and have the Weibull-H family of distribution as subfamily in order to achieve more flexibility and better fitting to the real life data, we considered the Modified Beta-H family of distributions.

The Beta- G family of distributions has also been explored by researchers after it was derived by Eugene et.al(2002) ([14]) and applied to the normal distribution. It has been used as generator to develop new distributions in order to achieve better fit for data and flexibility. Examples of these include Beta-Weibull distribution ([15]), Beta Half logistic distribution ([19]), Beta Type I generalized half logistic distribution ([8]), Beta Exponential distribution ([23]) e.t.c. Using the Beta-G family, some researchers were able to bring new families of distributions which include Beta Transmuted-G ([2]), Beta Weibull G ([21]), Beta-Odd Log-Logistic G ([10]) e.t.c. Nadarajah et.al(2014) ([24]) obtained a modified form of the beta G distribution called the Modified beta distribution. It has the c.d.f as

$$F(x;\sigma) = \int_0^{\frac{\theta H(x;\sigma)}{1+(\theta-1)H(x;\sigma)}} x^{\zeta-1}(1-x)^{\phi-1}dt \quad (3)$$

which equivalently gives

$$F(x;\sigma) = I_{\frac{\theta H(x;\sigma)}{1+(\theta-1)H(x;\sigma)}}(\zeta,\phi) = \frac{B(r;\zeta,\phi)}{B(\zeta,\phi)} \quad (4)$$

and the corresponding p.d.f as

$$f(x;\sigma) = \frac{\theta^{\zeta}\left[h(x;\sigma)(H(x;\sigma))^{\zeta-1}(1-H(x;\sigma))^{\phi-1}\right]}{B(\zeta,\phi)\left[1-(1-\theta)H(x;\sigma)\right]^{\zeta+\phi}} \quad (5)$$

where $r = \frac{\theta H(x;\sigma)}{1+(\theta-1)H(x;\sigma)}$,$B(\zeta,\phi) = \int_0^1 x^{\zeta-1}(1-x)^{\phi}\partial x$ and $B(r;\zeta,\phi)$ is an incomplete beta function.ζ, θ, and ϕ are shape parameters, $I_{\frac{\theta H(x;\sigma)}{1+((\theta-1)H(x;\sigma))}}(\zeta,\phi)$ is the incomplete beta function ratio. The additional parameters θ, ζ, and ϕ govern the skewness and tail weight of the generated distribution. An attractive feature of this family, just like the Beta G family, is that ζ and ϕ can afford greater control over the weights in both tails and in the center of the distribution. If ζ=ϕ=θ =1, we have the h(x;σ) and H(x;σ) of the baseline distribution.

The Modified Beta G has not been extensively used in obtaining generalized distributions. Some of its submodels in literature are Modified Beta Gompertz ([12]), Modified Beta Modified Weibull ([25]) e.t.c. Therefore, in this research, we are combining both the modified beta-G family of distributions and the Weibull-H family of distributions to obtain a new family of distributions that is more flexible and has the potentiality to fit data better than the modified beta-G and the Weibull-G family of distributions.

The plan of the paper is as follows. The Modified Beta Weibull G family of distributions is derived and defined in Section 2. Section 3 discusses the mixture representation of the p.d.f and c.d.f of the family of distributions. In Section 4, some statistical properties of this family of distributions were studied and discussed. Maximum Likelihood Estimation for the parameters of the family of distributions is discussed in Section 5. In Section 6,the family of distributions was applied to two real data sets.Section 7 has the concluding remarks given.

2. Derivation of the modified beta Weibull-G family of distribution

In this section, the p.d.f and c.d.f of the Modified Beta Weibull- G (MBWG) family of distributions are discussed. Inserting Equations 1 and 2 in Equation 5, we obtained the p.d.f of the MBWG as

$$f(x;\sigma) = \frac{\theta^{\zeta}\left[1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}\right]^{\zeta-1}\left[e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}\right]^{\phi-1}\alpha\rho g(x;\sigma)\frac{G(x;\sigma)^{\rho-1}}{S(x;\sigma)^{\rho+1}}e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}}{B(\zeta,\phi)\left[1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})\right]^{(\zeta+\phi)}} \quad x>0,\theta,\zeta,\alpha,\rho,\phi>0 \quad (6)$$

The corresponding c.d.f of MBWG is derived by inserting equation 1 in 3 to have

$$F(x;\sigma) = \int_0^{\frac{\theta(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}{1+(\theta-1)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}} x^{\zeta-1}(1-x)^{\phi-1}dt \quad (7)$$

Alternatively, equation 7 can be written as

$$F(x;\sigma) = I_{\frac{\theta(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}{1+(\theta-1)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}}(\zeta,\phi) = \frac{B(v;\zeta,\phi)}{B(\zeta,\phi)}$$

$$Lx>0,\theta,\zeta,\alpha,\rho,\phi>0 \quad (8)$$

where

$$v = \frac{\theta(1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}{1 + (\theta - 1)(1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})} \tag{9}$$

From Equation 8, the survival function of the MBWG is

$$1 - F(x;\sigma) = \frac{B(\zeta, \phi) - B(v; \zeta, \phi)}{B(\zeta, \phi)} \tag{10}$$

where v is written as in equation 9. The hazard function hz(x;σ) given as

$$hz(x;\sigma) = \frac{\theta^{\zeta}[1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}]^{\zeta-1}[e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}]^{\phi-1}\alpha\rho g(x;\sigma)\frac{G(x;\sigma)^{\rho-1}}{S(x;\sigma)^{\rho+1}}e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}}{(B(\zeta, \phi) - B(v; \zeta, \phi))[1 - (1 - \theta)(1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})]^{(\zeta+\phi)}}$$

and the reverse hazard function rhz(x;σ) as

$$rhz(x;\sigma) = \frac{\theta^{\zeta}[1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}]^{\zeta-1}[e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}]^{\phi-1}\alpha\rho g(x;\sigma)\frac{G(x;\sigma)^{\rho-1}}{S(x;\sigma)^{\rho+1}}e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}}{B(v; \zeta, \phi)[1 - (1 - \theta)(1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})]^{(\zeta+\phi)}}$$

2.1. *Sub-Family of the MBWG family of distributions*

Some of the sub-family of the MBWG family of distributions includes:

1. When θ=ζ=ϕ=1, we obtain the Weibull-G distribution of Bourguignon et.al(2014)
2. When ζ=ϕ=1, we have the Modified Weibull-G family of distribution(**NEW**) as

$$f(x;\sigma) = \frac{\theta\alpha\rho g(x;\sigma)\frac{G(x;\sigma)^{\rho-1}}{S(x;\sigma)^{\rho+1}}e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}}{[1 - (1 - \theta)(1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})]^{2}}$$

3. When θ=1, ϕ=1, we have the Exponentiated Weibull-G family of distributions by Cordeiro et.al(2013)
4. When ϕ = 1, we have the modified exponentiated Weibull G family of distributions **(NEW)** as

$$f(x;\sigma) = \frac{\zeta\theta^{\zeta}[1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}]^{\zeta-1}\alpha\rho g(x;\sigma)\frac{G(x;\sigma)^{\rho-1}}{S(x;\sigma)^{\rho+1}}e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}}{[1 - (1 - \theta)(1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})]^{-(\zeta+1)}}$$

5. When θ=ζ=ϕ=ρ=1, we have the Exponentiated-G family of distributions of Bareto-Souzza and Simas(2013)
6. When ϕ=ρ=1, we have the Modified Beta G family of distributions of Nadarajah et.al(2014)

2.2. *Submodels of the MBW-G family of distributions*

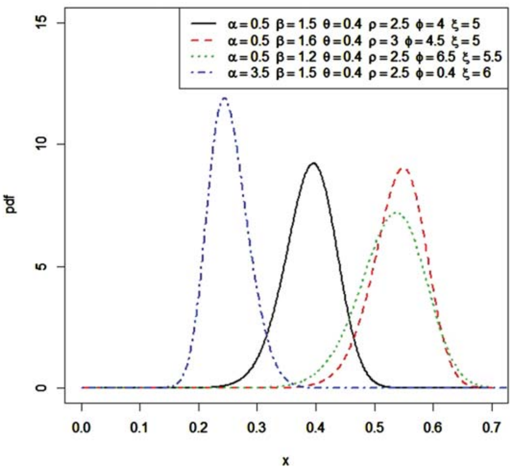
In this section, three(3) special models of the MBWG family of distributions are presented. These models generalize some models in literatures. The models have baselines of Exponential(Ex), Weibull(W) and Frechet(F) distributions.

2.3. *Modified Beta Weibull Exponential (MBWEx) distribution*

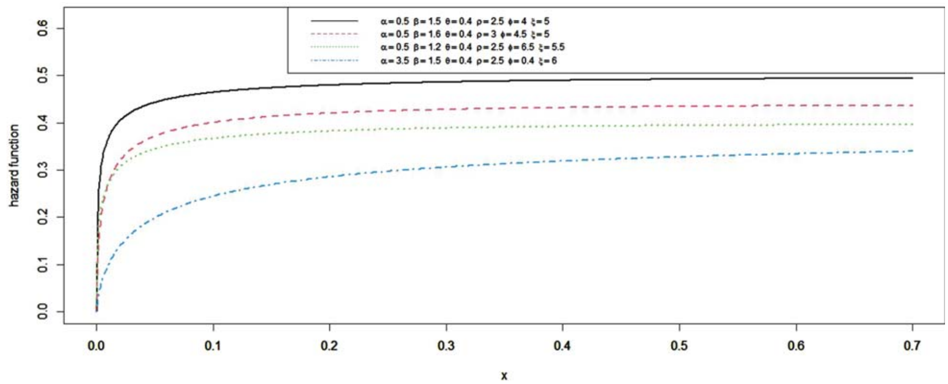
The pdf and cdf of exponential distribution are given as

$$g(x;\beta) = \beta e^{-\beta x}$$

$$G(x;\beta) = 1 - e^{-\beta x}$$



(a) pdf of the MBWEx.



(b) hf of the MBWEx.

Fig. 1. Plots of the Modified Beta Weibull exponential distribution.

Therefore, the pdf (f_{MBWEx}) and hazard function (h_{MBWEx}) of the MBWEx distribution is given as

$$f_{MBWEx} = \frac{\theta^{\zeta}[1-e^{-\alpha\left(\frac{(1-e^{-\beta x})}{(e^{-\beta x})}\right)^{\rho}}]^{\zeta-1}[e^{-\alpha\left(\frac{(1-e^{-\beta x})}{(e^{-\beta x})}\right)^{\rho}}]^{\phi-1}\alpha\rho\beta e^{-\beta x}\frac{(1-e^{-\beta x})^{\rho-1}}{(e^{-\beta x})^{\rho+1}}e^{-\alpha\left(\frac{(1-e^{-\beta x})}{(e^{-\beta x})}\right)^{\rho}}}{B(\zeta,\phi)[1-(1-\theta)(1-e^{-\alpha\left(\frac{(1-e^{-\beta x})}{(e^{-\beta x})}\right)^{\rho}})]^{(\zeta+\phi)}}$$

and

$$h_{MBWEx} = \frac{\theta^{\zeta}[1-e^{-\alpha\left(e^{\beta x}-1\right)^{\rho}}]^{\zeta-1}[e^{-\alpha\left(e^{\beta x}-1\right)^{\rho}}]^{\phi-1}\alpha\rho g(x;\sigma)\frac{(1-e^{-\beta x})^{\rho-1}}{(e^{-\beta x})^{\rho+1}}e^{-\alpha\left(e^{\beta x}-1\right)^{\rho}}}{B(\zeta,\phi)-B(v;\zeta,\phi)[1-(1-\theta)(1-e^{-\alpha(e^{\beta x}-1)^{\rho}})]^{-(\zeta+\phi)}}$$

respectively and x>0,θ,ζ,α,ρ,ϕ, $\beta > 0$.

The MBWEx distribution includes the Weibull Exponential(WE) when θ=ζ=ϕ=1. For θ=α=ρ=1, the MBWEx becomes Beta Exponential(BE) distribution. For θ=ζ=1, MBEx reduces to Exponentiated Weibull Exponential(EWE) distribution. Plots of the density function and the hazard function of the MBWEx with various assigned parameter values are shown in Fig. 1.

2.4. Modified Beta Weibull Weibull (MBWW) distribution

The pdf and cdf of Weibull distribution are given as

$$G(x;\beta,\lambda)=1-e^{-(\beta x)^{\lambda}}$$

$$g(x;\beta,\lambda)=\lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}$$

Now, the pdf f_{MBWW} and hazard function h_{MBWW} of the MBWW distribution is given as

$$f_{MBWW}=\frac{\theta^{\zeta}[1-e^{-\alpha\left(\frac{(1-e^{-(\beta x)^{\lambda}})}{(e^{-(\beta x)^{\lambda}})}\right)^{\rho}}]^{\zeta-1}[e^{-\alpha\left(\frac{(1-e^{-(\beta x)^{\lambda}})}{(e^{-(\beta x)^{\lambda}})}\right)^{\rho}}]^{\phi-1}\alpha\rho\lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\frac{(1-e^{-(\beta x)^{\lambda}})^{\rho-1}}{(e^{-(\beta x)^{\lambda}})^{\rho+1}}e^{-\alpha\left(\frac{(1-e^{-(\beta x)^{\lambda}})}{(e^{-(\beta x)^{\lambda}})}\right)^{\rho}}}{B(\zeta,\phi)[1-(1-\theta)(1-e^{-\alpha\left(\frac{(1-e^{-(\beta x)^{\lambda}})}{(e^{-(\beta x)^{\lambda}})}\right)^{\rho}})]^{(\zeta+\phi)}}$$

and

$$h_{MBWW}=\frac{\theta^{\zeta}[1-e^{-\alpha(e^{(\beta x)^{\lambda}}-1)^{\rho}}]^{\zeta-1}[e^{-\alpha(e^{(\beta x)^{\lambda}}-1)^{\rho}}]^{\phi-1}\alpha\rho\lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\frac{(1-e^{-(\beta x)^{\lambda}})^{\rho-1}}{(e^{-(\beta x)^{\lambda}})^{\rho+1}}e^{-\alpha(e^{(\beta x)^{\lambda}}-1)^{\rho}}}{(B(\zeta,\phi)-B(v;\zeta,\phi))[1-(1-\theta)(1-e^{-\alpha(e^{(\beta x)^{\lambda}}-1)^{\rho}})]^{(\zeta+\phi)}}$$

respectively, where x>0,θ,ζ,α,ρ,ϕ, β,$\lambda > 0$.

For θ=α=ρ=1, the MBWW becomes Beta Weibull(BW) distribution. For θ=ζ=1, MBWW reduces to Exponentiated Weibull Weibull(EWE) distribution. Plots of the density function and the hazard function of the MBWW with various assigned parameter values are shown in Fig. 2.

2.5. Modified Beta Weibull Frechet (MBWF) distribution

The pdf and cdf of Frechet distribution are given as

$$G(x;\beta,\lambda)=e^{-(\beta x+\lambda)}$$

$$g(x;\beta,\lambda)=\lambda\beta x^{-(\lambda+1)}e^{-(\beta x+\lambda)}$$

Now, the pdf of the MBWF f_{MBWF} distribution is given as

$$f_{MBWF}=\frac{\theta^{\zeta}[1-e^{-\alpha\left(\frac{(e^{-(\beta x+\lambda)})}{(1-e^{-(\beta x+\lambda)})}\right)^{\rho}}]^{\zeta-1}[e^{-\alpha\left(\frac{(e^{-(\beta x+\lambda)})}{(1-e^{-(\beta x+\lambda)})}\right)^{\rho}}]^{\phi-1}\alpha\rho\lambda\beta x^{-(\lambda+1)}e^{-(\beta x+\lambda)}\frac{(e^{-(\beta x+\lambda)})^{\rho-1}}{(1-e^{-(\beta x+\lambda)})^{\rho+1}}e^{-\alpha\left(\frac{(e^{-(\beta x+\lambda)})}{1-e^{-(\beta x+\lambda)}}\right)^{\rho}}}{B(\zeta,\phi)[1-(1-\theta)(1-e^{-\alpha\left(\frac{(e^{-(\beta x+\lambda)})}{(1-e^{-(\beta x+\lambda)})}\right)^{\rho}})]^{(\zeta+\phi)}}$$

with the hazard function h_{MBWF}

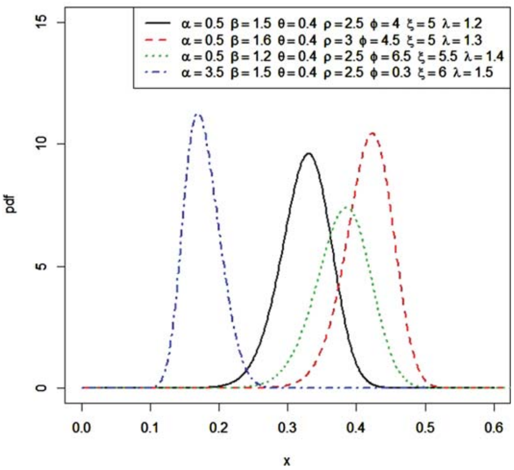
$$h_{MBWF}=\frac{\theta^{\zeta}[1-e^{-\alpha\left(\frac{e^{-(\beta x+\lambda)}}{1-e^{-(\beta x+\lambda)}}\right)^{\rho}}]^{\zeta-1}[e^{-\alpha\left(\frac{e^{-(\beta x+\lambda)}}{1-e^{-(\beta x+\lambda)})}\right)^{\rho}}]^{\phi-1}\alpha\rho\lambda\beta x^{-(\lambda+1)}e^{-(\beta x+\lambda)}\frac{(e^{-(\beta x+\lambda)})^{\rho-1}}{(1-e^{-(\beta x+\lambda)})^{\rho+1}}e^{-\alpha\left(\frac{e^{-(\beta x+\lambda)}}{1-e^{-(\beta x+\lambda)}}\right)^{\rho}}}{(B(\zeta,\phi)-B(v;\zeta,\phi))[1-(1-\theta)(1-e^{-\alpha\left(\frac{e^{-(\beta x+\lambda)}}{1-e^{-(\beta x+\lambda)}}\right)^{\rho}})]^{(\zeta+\phi)}}$$

respectively where $x > 0, \theta, \zeta, \alpha, \rho, \phi, \beta, \lambda > 0$

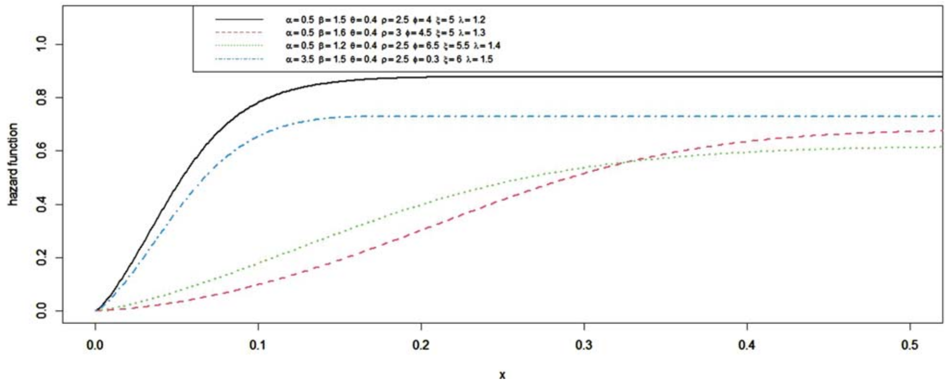
The MBWF distribution includes the Weibull Frechet(WF) when θ=ζ=ϕ=1. For θ=α=ρ=1, the MBWF becomes Beta Frechet(BF) distribution. For θ=ζ=1. Plots of the density function and the hazard function of the MBWF with various assigned parameter values are shown in Fig. 3.

3. Linear representation

In this section, a useful mixture representation for the density function of the MBWG distribution is derived. The derived representation is crucial for the derivation of statistical properties of the MBWG distribution such as moments, generating functions, order statistic properties.



(a) pdf of the MBWW.



(b) hf of the MBWW.

Fig. 2. Plots of the Modified Beta Weibull Weibull distribution.

Using the Binomial Expression given as

$$(1-z)^{b-1} = \sum_{p=0}^{\infty}(-1)^p \binom{b}{p} z^p \tag{11}$$

where $|z| < 1$ and b>0, Therefore,

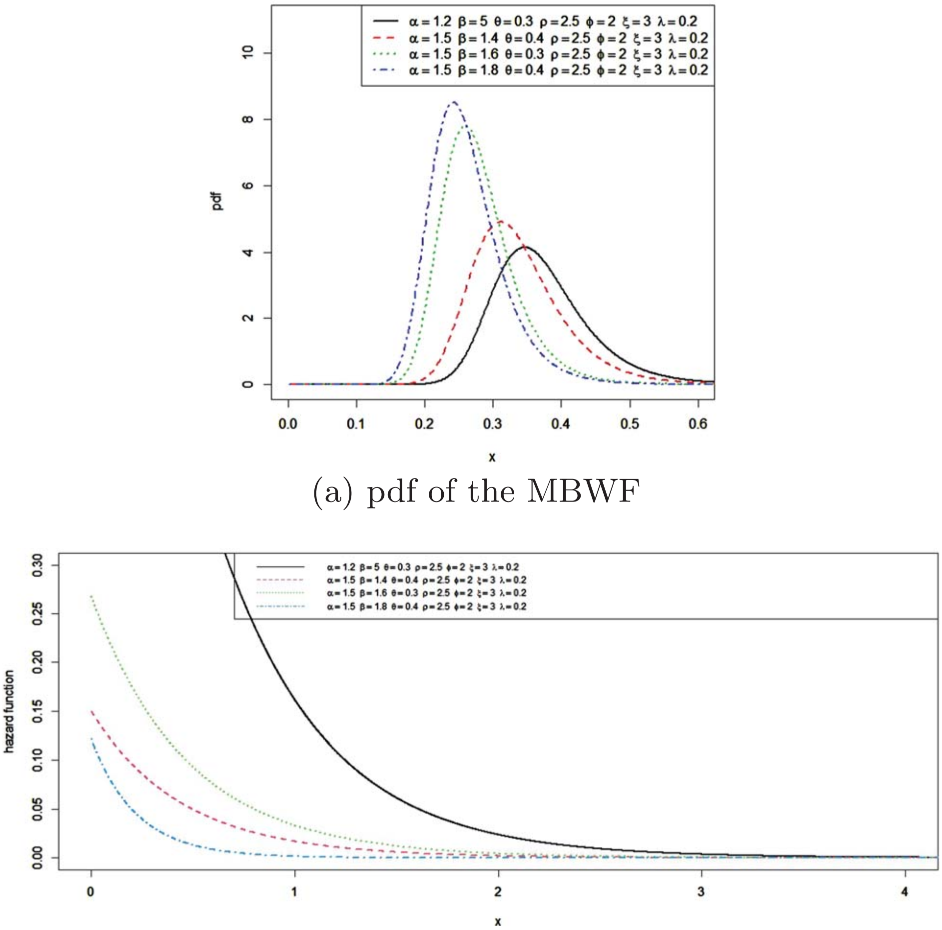
$$[1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})]^{-(\zeta+\phi)} = \sum_{w=0}^{\infty}(-1)^w \binom{-\zeta-\rho}{w}(1-\theta)^w(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})^w \tag{12}$$

and

$$(1-(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}))^{\phi-1} = \sum_{s=0}^{\infty}(-1)^s \binom{\zeta-1}{s}(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})^s \tag{13}$$

Inserting the expressions in Equations 12 and 13 in the p.d.f of the MBWG family as in Equation 5, we obtain

$$f(x;\sigma) = \theta^{\zeta}\alpha\rho \sum_{s=0}^{\infty}\sum_{w=0}^{\infty}(-1)^{w+s}\binom{\phi-1}{s}\binom{-\zeta-\phi}{w}(1-\theta)^w(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})^{w+s+\phi-1}$$



(a) pdf of the MBWF

(b) hf of the MBWF

Fig. 3. Plot of pdf of the Modified Beta Weibull Frechet distribution.

$$g(x;\sigma)\frac{G(x;\sigma)^{\rho-1}}{S(x;\sigma)^{\rho+1}}e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}} \tag{14}$$

Writing expression in Equation 14 in terms of Exponentiated Weibull-G, we have the p.d.f as

$$f(x;\sigma) = \sum_{s=0}^{\infty}\sum_{w=0}^{\infty} r_{ij}\Pi_{w+s+\phi} \tag{15}$$

where

$$r_{ij} = \frac{(-1)^{w+s}\binom{\phi-1}{s}\binom{-\zeta-\phi}{w}(1-\theta)^{w}\theta^{\zeta}\alpha\rho}{w+s+\phi}$$

and

$$\Pi_{w+s+\phi} = (w+s+\phi)g(x;\sigma)\frac{G(x;\sigma)^{\rho-1}}{S(x;\sigma)^{\rho+1}}e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})^{w+s+\phi-1}$$

$\Pi_{w+s+\phi}$ is the p.d.f of the Exponentiated Weibull-G family of distributions with power parameters(w+s+ϕ). Further simplifying equation 14,

$$(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})^{w+s+\phi-1}=\sum_{l=0}^{\infty}(-1)^{l}\binom{w+s+\phi-1}{l}(e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})^{l} \tag{16}$$

Inserting Equation 16 in Equation 14 we have

$$f(x;\sigma)=\theta^{\zeta}\alpha\rho g(x;\sigma)\sum_{s=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{w+s+l}(1-\theta)^{w}\binom{\phi-1}{s}\binom{-\zeta-\phi}{w}\binom{w+s+\phi-1}{l}(e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})^{w+s+l+\phi-1} \tag{17}$$

Re-writing

$$(e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})^{w+s+l+\phi-1}=\sum_{d=0}^{\infty}(-1)^{d}\frac{[\alpha(w+s+l+\phi-1)]^{d}}{d!}\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho d} \tag{18}$$

Therefore,inserting Equation 18 in 17 we have the expression for the p.d.f of the MBWG distribution as

$$f(x;\sigma)=\theta^{\zeta}\alpha\rho g(x;\sigma)\sum_{s=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}(-1)^{w+s+l+d}(1-\theta)^{w}\binom{\phi-1}{s}\binom{-\zeta-\phi}{w}\binom{w+s+\phi-1}{l}$$

$$\frac{[\alpha(w+s+l+\zeta-1)]^{d}}{d!}\frac{G(x;\sigma)^{w+s+\phi+d-1}}{S(x;\sigma)^{w+s+\phi+d+1}} \tag{19}$$

Note that

$$S(x;\sigma)^{w+s+\phi+d+1}=(1-G(x;\sigma))^{w+s+\phi+d+1} \tag{20}$$

Using the binomial expression in 20, we have the expression as

$$(1-G(x;\sigma))^{w+s+\phi+d+1}=\sum_{k=0}^{\infty}(-1)^{k}\binom{w+s+\phi+d}{k}G(x;\sigma)^{k} \tag{21}$$

Inserting Equation 21 in 19, we have

$$f(x;\sigma)=\sum_{s=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty}\theta^{\zeta}\alpha\rho g(x;\sigma)(-1)^{w+s+l+d+k}(1-\theta)^{w}\binom{\phi-1}{s}\binom{-\zeta-\phi}{w}\binom{w+s+\phi-1}{l}$$

$$\binom{w+s+\phi+d}{k}\frac{[\alpha(w+s+l+\zeta-1)]^{d}}{d!}G(x;\sigma)^{w+s+\phi+d+k-1} \tag{22}$$

Expressing Equation 22 in terms of Exp-G distribution, we have the p.d.f as

$$f(x;\sigma)=\sum_{s=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty}r_{w,s,l,d}\Theta_{w+s+\phi+d+k} \tag{23}$$

where

$$r_{w,s,l,d}=\theta^{\zeta}\alpha\rho(-1)^{w+s+l+d+k}(1-\theta)^{w}\binom{\phi-1}{s}\binom{-\zeta-\phi}{w}\binom{w+s+\phi-1}{l}\binom{w+s+\phi+d}{k}$$

$$\frac{[\alpha(w+s+l+\zeta-1)]^{d}}{d!w+s+\phi+d+k}$$

and

$$\Theta_{w+s+\phi+d+k} = (w+s+\phi+d+k)g(x;\sigma)G(x;\sigma)^{w+s+\phi+d+k-1}$$

$\Theta_{w+s+\phi+d+k}$ is the p.d.f of the Exponentiated-G family of distributions. Expressions obtained in this section find its use when studying properties of the distribution such as moments, order statistics e.t.c.
Furthermore, integrating expression in 23, we have the linear mixture c.d.f of the MBWG family in terms of the Exp-G distribution as

$$F(x;\sigma) = \sum_{s=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty} r_{w,s,l,d}\, Q_{w+s+\phi+d+k} \tag{24}$$

where $Q_{w+s+\phi+d+k}$ is the c.d.f of the Exponentiated-G family of distribution

4. Statistical properties

In this section, the statistical properties of the MBWG distribution are extensively studied. The properties considered are moments, monet generating function, quantile function, and order statistics

4.1. Quantile function

The quantile function of the MBWG distribution is obtained by solving

$$I_{\frac{\theta(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}{1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}}(\zeta,\phi) = u \tag{25}$$

where u is uniform variate on the interval [0,1]. Therefore,

$$\frac{\theta(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}{1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})} = \Pi$$

$$\theta(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}) = \Pi(1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}))$$

$$\theta(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}) + \Pi(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}) = \Pi$$

$$1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}(\theta+\Pi(1-\theta)) = \Pi$$

$$1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}} = \frac{\Pi}{\theta+\Pi(1-\theta)}$$

$$e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}} = 1-\frac{\Pi}{\theta+\Pi(1-\theta)}$$

$$\frac{G(x;\sigma)}{S(x;\sigma)} = \frac{G(x;\sigma)}{1-G(x;\sigma)} = (-\frac{1}{\alpha}log(1-\frac{\Pi}{\theta+\Pi(1-\theta)}))^{\frac{1}{\rho}}$$

$$G(x;\sigma) = \frac{(-\frac{1}{\alpha}log(1-\frac{\Pi}{\theta+\Pi(1-\theta)}))^{\frac{1}{\rho}}}{1+((-\frac{1}{\alpha}log(1-\frac{\Pi}{\theta+\Pi(1-\theta)}))^{\frac{1}{\rho}})}$$

where

$$\Pi = I_u^{-1}(\zeta, \phi)$$

Π is the inverse of the incomplete beta function ratio.

4.2. Moments

The r^{th} moment μ_r' of a distribution is given as

$$E[X^r] = \int_0^\infty x^r f(x;\sigma) \tag{26}$$

Alternatively, using the mixture expression as in equation 23, the r^{th} moment of the distribution is

$$E[X^r] = \sum_{s=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty} r_{w,s,l,d} E[\Theta^r_{w+s+\phi+d+k}]$$

where $\Pi_{w+s+\phi+d+k}$ is the Exp-G with power parameters w+s+ϕ+d+k.
For the n^{th} central moment of X (M_n), the expression is obtained as

$$M_n = E[(X-\mu')^r] = \sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}(\mu')^{n-r}E[X^r] \tag{27}$$

Inserting the mixture expression of 23 in 27, the expression for the r^{th} moment becomes

$$M_n = \sum_{r=0}^{n}\sum_{s=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{r}(-1)^{n-r}(\mu')^{n-r} r_{w,s,l,d} E[\Theta^r_{w+s+\phi+d+k}] \tag{28}$$

The cumulants, η_n, of X_n follow recursively from

$$\eta_n = \mu_n' - \sum_{r=0}^{n-1}\binom{n-1}{r-1}\eta_r\mu_{n-r}' \tag{29}$$

where η_1= μ_1', η_2=$\mu_2' - (\mu_1')^2$, η_3=μ_3'-3$\mu_2'\mu_1'$+$(\mu_1')^3$. From these expressions, the skewness and kurtosis properties of the family of distribution can be measured from the ordinary moments.

4.3. Moment generating function

The moment generating function ($M_f(t)$)= E[e^{tX}] of X.In this section, two formula in computing the moment generating function(m.g.f) of the family is discussed.
Firstly, we derived the m.g.f form equation 23 as

$$M_f(t) = \sum_{s=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty} r_{w,s,l,d} M_{w+s+\phi+d+k}$$

where $M_{w+s+\phi+d+k}$ is the m.g.f of the Exp-G with power parameter w+s+ϕ+d+k.
Secondly, the m.g.f was also derived from the baseline of the quantile function as

$$M_f(t) = \sum_{s=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty}(w+s+\phi+d+k) r_{w,s,l,d} D_{w+s+\phi+d+k}$$

where $D_{w+s+\phi+d+k}$= $\int_0^1 e^{tq_f(u)} u^{w+s+\phi+d+k-1}$∂u and $q_f(u)$ is the quantile function corresponding to f(x;σ).

4.4. Order statistics

Let $X_1, ..., X_n$ be a random sample from the MBWG distribution, The pdf of i^{th} order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x_i;\beta) = \frac{1}{B(i, n-i+1)} f(x_i;\beta) \sum_{v=0}^{n-1} (-1)^v \binom{n-1}{v} F(x_i;\beta)^{v+i-1} \tag{30}$$

Therefore inserting the p.d.f as in equation 23 and c.d.f as in equation 24 in 30, we have the order statistics of MBWG as

$$f_{i:n}(x_i;\beta) = \frac{1}{B(i, n-i+1)} \sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty} r_{w,s,l,d}\Theta_{w+s+\phi+d+k} \sum_{v=0}^{n-1} (-1)^v \binom{n-1}{v}$$

$$\left(\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty} r_{w,s,l,d} Q_{w+s+\phi+d+k}\right)^{v+i-1} \tag{31}$$

Using an equation given in page 17 of Gradshteyn and Ryzhik ([27]) for a power series raised to a positive integer n, then equation 31 becomes

$$f_{i:n}(x_i;\beta) = \frac{1}{B(i, n-i+1)} \sum_{v=0}^{n-i} (-1)^v \binom{n-i}{v} \sum_{a=0}^{\infty} T_{r,v+i-1}\Theta_r$$

where r= w+s+ϕ+d+k and Θ_r is the pdf of the Exp-G with power parameter r

5. Parameter estimation

In this section, we derived the expressions for the estimates of the parameters of the MBWG distribution. We employed the use of the maximum likelihood estimation (M.L.E) method. Let δ=(ζ,ϕ,θ,α,ρ,σ) be the parameter vector and $x = (x_1, ..., x_n)$ be the sample from the MBWG distribution, then the log-likelihood function for δ can be written as

$$l(\delta) = n\zeta \ln\theta - n\ln B(\zeta,\phi) + (\zeta-1)\sum_{i=1}^{n} \ln(1 - e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}) - \alpha(\rho-1)\sum_{i=1}^{n}\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}$$

$$-(\zeta+\rho)\sum_{i=1}^{n} \ln(1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})^{\rho}) + n\ln\alpha + n\ln\rho + n\ln g(x;\sigma) + (\rho-1)\sum_{i=1}^{n}\ln G(x;\sigma)$$

$$-(\rho+1)\sum_{i=1}^{n}\ln S(x;\sigma) - \alpha\sum_{i=1}^{n}\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho} \tag{32}$$

The elements of the score vector, that is, the partial derivatives of l(δ) with respect to the parameters are

$$\frac{\partial l(\delta)}{\partial\zeta} = n\ln\theta - n\frac{B'(\zeta,\phi)}{B(\zeta,\phi)} + \sum_{i=1}^{n}\ln(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}) - \sum_{i=1}^{n}\ln(1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}) \tag{33}$$

$$\frac{\partial l(\delta)}{\partial\phi} = -n\frac{B'(\zeta,\phi)}{B(\zeta,\phi)} - \alpha\sum_{i=1}^{n}\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho} - \sum_{i=1}^{n}\ln(1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}) \tag{34}$$

$$\frac{\partial l(\delta)}{\partial \theta} = \frac{n\zeta}{\theta} - (\zeta+\phi)\sum_{i=1}^{n} \frac{1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}}{1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})} \tag{35}$$

$$\frac{\partial l(\delta)}{\partial \alpha} = (\zeta-1)\sum_{i=1}^{n} \frac{e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}{1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}} - (\phi-1)\sum_{i=1}^{n}\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho} + (\zeta+\beta)\sum_{i=1}^{n} \frac{(1-\theta)\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho} e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}}{1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}$$

$$+\frac{n}{\alpha} - \sum_{i=1}^{n}\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho} \tag{36}$$

$$\frac{\partial l(\delta)}{\partial \rho} = (\zeta-1)\sum_{i=1}^{n} \frac{\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}\ln\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}}{1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}} - (\phi-1)\sum_{i=1}^{n} \frac{\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}\ln\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)}{e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}} + \frac{n}{\rho} + \ln\sum_{i=1}^{n} G(x;\sigma)$$

$$-(\zeta+\phi)\sum_{i=1}^{n} \frac{\alpha(1-\theta\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}\ln\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right))e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}}}{1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})} - \sum_{i=1}^{n} S(x;\sigma) - \alpha\sum_{i=1}^{n}\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}\ln\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right) \tag{37}$$

$$\frac{\partial l(\delta)}{\partial \sigma} = (\zeta-1)\sum_{i=1}^{n}\alpha\rho\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho-1}\frac{G'(x;\sigma)}{S^2(x;\sigma)} - (\phi-1)\sum_{i=1}^{n}\alpha\rho\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho-1}\frac{G'(x;\sigma)}{S^2(x;\sigma)} + \sum_{i=1}^{n}\frac{g'(x;\sigma)}{g(x;\sigma)}$$

$$+(\rho-1)\sum_{i=1}^{n}\frac{G'(x;\sigma)}{G(x;\sigma)} + (\rho+1)\sum_{i=1}^{n}\frac{G'(x;\sigma)}{S(x;\sigma)} + (\zeta+\phi)\sum_{i=1}^{n}\frac{(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})\alpha\rho\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho-1}\frac{G'(x;\sigma)}{S^2(x;\sigma)}}{1-(1-\theta)(1-e^{-\alpha\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho}})}$$

$$-\rho\alpha\sum_{i=1}^{n}\left(\frac{G(x;\sigma)}{S(x;\sigma)}\right)^{\rho-1}\frac{G'(x;\sigma)}{S^2(x;\sigma)} \tag{38}$$

Setting the set of equations in 33–38, to be equals to zero and solving them simultaneously yields the MLE $\hat{\delta}=(\hat{\zeta}, \hat{\phi},\hat{\theta},\hat{\alpha},\hat{\rho},\hat{\sigma})$ of $\delta=(\zeta,\phi,\theta,\alpha,\rho,\sigma)$. Solving these equations cannot be done analytically. This can be achieved by the aid of statistical software using iterative methods such as Newton-Raphson type algorithms to solve numerically.

For interval estimation of the model parameters, we require the observed information matrix for interval estimation and test of hypothesis on the parameters $(\zeta, \phi, \theta, \alpha, \rho, \theta)$, we obtain a 6x6 unit information matrix

$$J = \begin{bmatrix} J_{\zeta,\zeta} & J_{\zeta,\phi} & J_{\zeta,\theta} & J_{\zeta,\alpha} & J_{\zeta,\rho} & J_{\zeta,\sigma} \\ J_{\zeta,\phi} & J_{\phi,\phi} & J_{\phi,\theta} & J_{\phi,\alpha} & J_{\phi,\rho} & J_{\phi,\sigma} \\ J_{\zeta,\theta} & J_{\phi,\theta} & J_{\theta,\theta} & J_{\theta,\alpha} & J_{\theta,\rho} & J_{\theta,\sigma} \\ J_{\zeta,\alpha} & J_{\alpha,\phi} & J_{\alpha,\theta} & J_{\alpha,\alpha} & J_{\alpha,\rho} & J_{\alpha,\sigma} \\ J_{\zeta,\rho} & J_{\rho,\phi} & J_{\rho,\theta} & J_{\rho,\alpha} & J_{\rho,\rho} & J_{\rho,\sigma} \\ J_{\zeta,\sigma} & J_{\sigma,\phi} & J_{\sigma,\theta} & J_{\sigma,\alpha} & J_{\sigma,\rho} & J_{\sigma,\sigma} \end{bmatrix}$$

The corresponding elements are derived by the second derivatives of l with respect to the parameters.

Under the conditions that are fulfilled for parameters, the asymptotic distribution of $\sqrt{n}(\hat{\delta}-\delta)$ is $N_6(0, J(\hat{\delta})^{-1})$ distribution of δ can be used to construct approximate confidence intervals and confidence regions for the parameters and for the hazard and survival functions. The asymptotic normality is also useful for testing the goodness of fit of the beta type I generalized half logistic distribution and for comparing this distribution with some of its special sub-models using one of these two well-known asymptotically equivalent test statistics namely, the likelihood ratio statistic and Wald statistic. An asymptotic confidence interval with significance level τ, for each parameter δ_i is given by

$$ACI(\delta_i, 100(1-\tau)) = \hat{\delta} - z_{\frac{\tau}{2}}\sqrt{J^{\hat{\delta},\hat{\delta}}}, \delta + z_{\frac{\delta}{2}}\sqrt{J^{\hat{\delta},\hat{\delta}}} \quad (39)$$

where $J^{\hat{\delta},\hat{\delta}}$ is the i^th diagonal element of $K_n(\hat{\delta})^{-1}$ for $i = 1, 2, 3, 4, 5, 6$ and $z_{\tau/2}$ is the quantile of the standard normal distribution.

6. Application to real life data sets

In this section, application of the MBWG distribution was done on two real datasets using the Weibull distribution as the baseline model to illustrate the importance and fit of the MBWG distribution. The maximum likelihood estimates (M.L.E) of the distribution and that of the competitive distributions will be obtained. We assessed the goodness of fit of the distributions using the log-likelihood, Akaike's information criterion (AIC), Bayesian information criterion (BIC), and corrected Akaike's information criterion (CAIC). The R statistical software is employed for data analysis. Estimation of model parameter estimates was done using the optim() function in stats packages in R version 4.1.1 [32]. The fit of the MBWW distribution is compared with other competitive distributions which are Weibull exponential (WEx) ([30]), Kumaraswamy Weibull (KW) [31]), Beta Weibull (BW) ([28]) and Exponentiated Weibull Weibull (ExWW) ([29]) distributions. The p.d.fs of these distributions are as follows:

- Weibull exponential (WEx) distribution.

$$f(x) = \alpha\lambda\beta(1 - e^{(-\beta x)})^{\lambda-1}e^{\lambda\beta x-\alpha(e^{\beta x}-1)^{\lambda}}$$

- Kumaraswamy Weibull (KW) distribution

$$f(x) = \alpha\tau\lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}$$
$$(1 - e^{-(\beta x)^{\lambda}})^{\tau-1}(1 - (1 - e^{-(\beta x)^{\lambda}})^{\tau})^{\alpha-1}$$

- Beta Weibull (BW) distribution

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}\left[1 - e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right]^{\alpha-1}e^{-\beta\left(\frac{x}{\lambda}\right)^{\theta}}$$

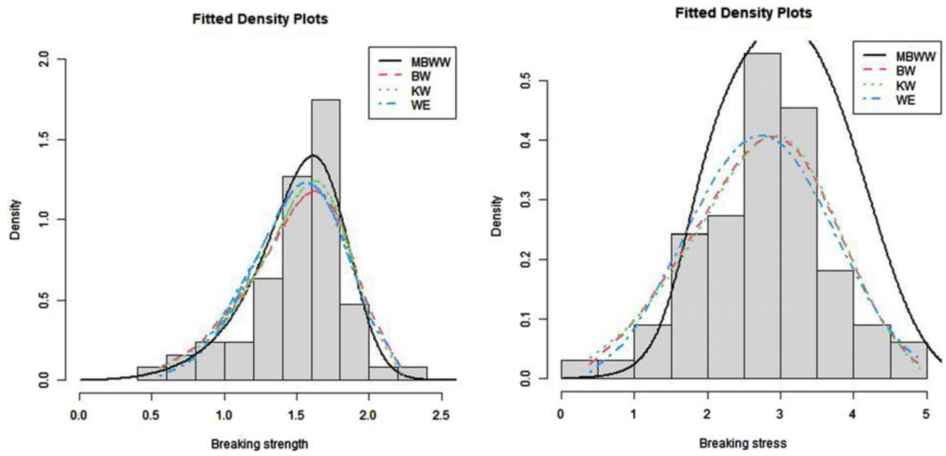
- Exponentiated Weibull Weibull (ExWW) distribution

$$f(x) = \alpha\theta\lambda\beta(e^{\beta x} - 1)^{\lambda-1}e^{[-(\alpha(e^{\beta x}-1)^{\lambda})-\beta x]}$$
$$[1 - e^{-\alpha(e^{\beta x}-1)^{\lambda}}]^{\theta-1}$$

Data set 1: The first data set represents the breaking strength of 100 yarn as reported by Gomes-Silva et al (2017). The data-set consists of 63 measurements of the strengths of 1.5 cm glass fibres, which were initially collected by United Kingdom National Physical Laboratory staff. The data is presented below:
0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28,1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50,1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58,1.59, 1.60, 1.61, 1.63,1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68,1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82,1.84, 1.84, 2.00, 2.01, 2.24.

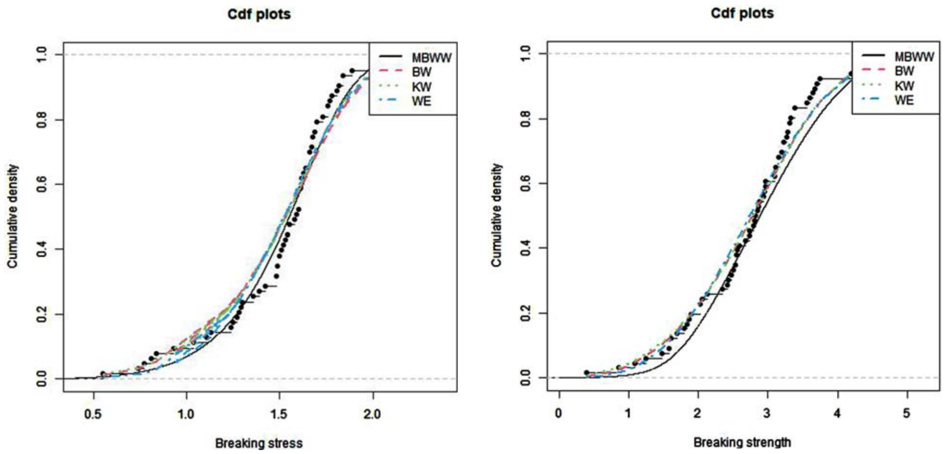
Data set 2:
The second data set represents the breaking stress of carbon fibers of 50 mm length (GPa) which was reported by Nicholas and Padgett (2006). This data was used by Yousof et al. (2017). The data set is:
0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

After analysis of the two datasets, Fig. 6 describes the shape of the hazard plots of the data. It shows that the hazard curve os non-decreasing for the two datasets. Furthermore Fig. 4 is the histogram which further reveals the fit of the model to both datasets. Figure 5 shows how close the fitted pdf is to the empirical distribution which further established the fact that the model fits the data well. In Tables 1 and 2, we have observed that the modified beta weibull weibull distribution gives the best fit when compared to its submodels, therefore making it the preferred model to consider for this data on the basis of the selection criterion considered.



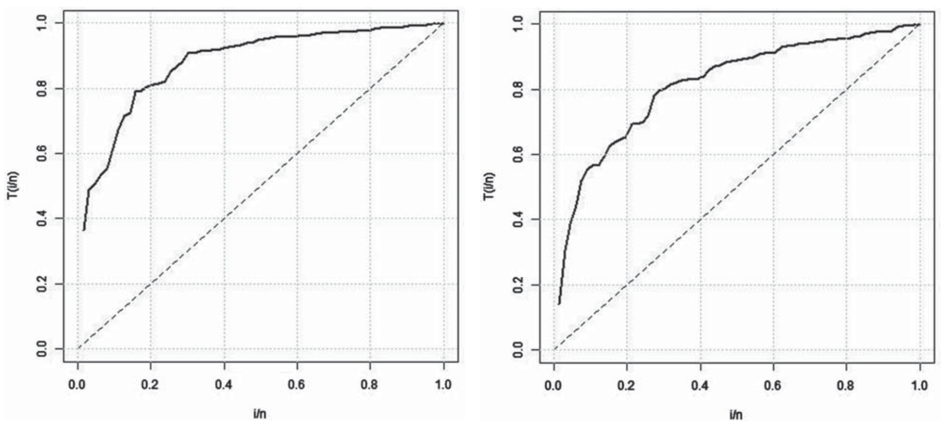
(a) Estimated pdf plot for the MBWW Data I (b) Estimated pdf plot for the MBWW Data II

Fig. 4. Estimated pdf plots for Data I and Data II.



(a) Estimated cdf plot for the MBWW for Data I (b) Estimated cdf plot for the MBWW for Data II

Fig. 5. Estimated cdf plot for the MBWW



(a) TTT plot of Breaking Strength of Yarn (b) TTT plot of Breaking Stress of Carbon Fibre

Fig. 6. TTT plot of MBWW

Table 1
The MLEs and Information Criteria of the models based on data set 1

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\rho}$	$\hat{\zeta}$	$\hat{\phi}$	-LL	AIC	BIC	CAIC
MBWW	1.5549	0.5758	0.7992	0.1928	2.2280	1.4243	1.5953	11.1390	36.2780	51.2799	38.9446
WEx	5.3583	1.4233	0.8726	-	-	-	-	15.5250	37.0500	43.4794	37.45684
KW	0.5040	0.1729	6.5245	1.2858	-	-	-	15.0980	38.1959	46.7685	38.8856
BW	0.6243	6.4974	7.1824	2.3007	-	-	-	14.6656	37.3311	45.9036	38.3837
EKW	1.8536	5.5418	0.4837	5.0944	2.0662	-	-	14.8422	39.6844	50.4001	40.7371

Table 2
The MLEs and Information Criteria of the models based on data set 2

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\rho}$	$\hat{\zeta}$	$\hat{\phi}$	-LL	AIC	BIC	CAIC
MBWW	0.6931	0.0250	1.4581	1.2725	0.7170	0.0823	1.5192	67.2027	148.4056	153.345	149.708
WEx	3.2187	1.4841	0.4835	-	-	-	-	86.3347	178.6695	185.2385	179.0566
KW	0.5380	0.1338	3.6983	1.8507	-	-	-	86.3365	180.6732	189.4318	181.3289
BW	0.7280	4.5270	3.9104	4.9847	-	-	-	86.1975	180.3951	189.1537	181.0508
EKW	0.0673	0.1641	2.9947	2.8437	1.4215	-	-	85.0870	180.1732	191.1214	181.1732

7. Conclusion

In applications to real life situations, there has always been a clear need for extended forms of existing distributions which are more flexible and gives better fit to model real data which has high degree of kurtosis and skewness. In this work, we proposed a new family of distribution called the Modified Beta Weibull Family which generalizes the beta weibull G family by the addition of a shape parameter. Some well known family of distributions are special cases of the Modified Beta Weibull Family. Some mathematical properties of the new class including linear expression of the distribution, ,moments,quantile, moment generating functions and order statistics are provided. The model parameters are estimated by the maximum likelihood estimation method and the observed information matrix is determined. We prove empirically by means of an application to a real data set that special cases of the proposed family can give better fits than other models generated from well-known families.

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