

A Discrete Competitive Facility Location Model with Minimal Market Share Constraints and Equity-Based Ties Breaking Rule

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Abstract. We consider a geographical region with spatially separated customers, whose demand is currently served by some pre-existing facilities owned by different firms. An entering firm wants to compete for this market locating some new facilities. Trying to guarantee a future satisfactory captured demand for each new facility, the firm imposes a constraint over its possible locations (a finite set of candidates): *a new facility will be opened only if a minimal market share is captured in the short-term*. To check that, it is necessary to know the exact captured demand by each new facility. It is supposed that customers follow the partially binary choice rule to satisfy its demand. If there are several new facilities with maximal attraction for a customer, we consider that the proportion of demand captured by the entering firm will be equally distributed among such facilities (equity-based rule). This ties breaking rule involves that we will deal with a nonlinear constrained discrete competitive facility location problem. Moreover, minimal attraction conditions for customers and distances approximated by intervals have been incorporated to deal with a more realistic model. To solve this nonlinear model, we first linearize the model, which allows to solve small size problems because of its complexity, and then, for bigger size problems, a heuristic algorithm is proposed, which could also be used to solve other constrained problems.

Key words: location, competitive model, heuristic algorithms.

1. Introduction

We consider a certain geographical area in which customers are supposed to be concentrated in demand points (see Francis *et al.*, 2002). The demand of these customers is fixed and currently satisfied by a set of facilities already established in that area belonging to one or several firms. When a new firm wants to enter this market with the idea of capturing as much demand as possible, the company is facing a problem of Competitive Location. The entering firm must decide where to locate its own facilities in order to maximize its total profit, which will depend on the total captured demand. This decision implies the

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study of the conditions that surround the distribution of a specific product, such as the location space, the characteristics of the facilities (quality), the customer choice rule, and additional constraints imposed by the firm, or by the local administration. The combination of all these factors will give rise to different competitive location models that will need a different study and the use of different techniques to solve them (see Eiselt *et al.*, 1993, 2015; Friesz *et al.*, 1988; Plastria, 2001; Reville and Eiselt, 2005; Al-Yakoob and Sherali, 2018).

A crucial issue in this type of models is to study the customers behaviour when choosing the facility or facilities that will satisfy their demand. In the literature, it has been usual to consider the distance to the facility as the first criterion used by the customer, so that the demand is served by the closest facility (see Drezner, 1995; Hotelling, 1929). Distance gives a first measure of attraction that each customer feels for each facility. Subsequently, the way of evaluating the attraction of customers by the facilities was modified, considering that facilities at the same distance did not have to be equally attractive for a customer if those facilities had different qualities (their size, number of parking spaces, leisure areas, etc). Different techniques emerge to measure the attraction between customers and facilities, but all of them have in common that this attraction is directly proportional to the quality of the facility, and inversely proportional to some function of the distance between them (Hodgson, 1978; Wilson, 1976). In this way, now the customer will be served by the most attractive facility, which would coincide with the closest one if all the facilities have the same quality. It was Huff (1964) who proposed for the first time a location model in which customers satisfied their demand according to a rule of proportional behaviour: customer demand will be split between all the facilities proportionally to the attraction it feels for each one.

The most common customer choice rules are then the *binary* or *deterministic* rule (the full demand of a customer is satisfied by only one facility, the one to which he is attracted most – *binary attraction*), and the *Huff*, *probabilistic* or *proportional* rule (a customer splits his demand over all facilities in the market proportionally with his attraction to each facility – *additive attraction*), see Drezner and Drezner (2004), Serra *et al.* (1999b). Although in many cases customers patronize facilities according to these two rules, there are others in which these rules do not represent customer behaviour properly. In this paper we are going to consider that customers follow the partially binary or multideterministic rule, where it is supposed that several firms are present in the market with some pre-existing facilities, and the full demand of a customer is served by all the firms but patronizing only one facility from each firm, the facility with the maximum attraction, then its demand is split between those facilities proportionally with their attraction (see Fernández *et al.*, 2017; Hakimi, 1990; Suárez-Vega *et al.*, 2004).

This unusual rule explains the customers behaviour in some everyday cases better. For example, suppose that to do his weekly shopping, a customer has several supermarkets around his home belonging to two different firms. Most likely, the customer does not do all the weekly shopping in a single supermarket, because there may be products that are not available in the supermarkets of one of the firms, or that their prices are higher than its competitor's. Nor the customer will go to all supermarkets of the same firm, since it is

assumed that all of them offer the same products at the same prices. Then, the customer will go to one of the supermarkets of each firm, the one with maximum attraction, and will buy more or less products in each supermarket depending of its attraction.

In our model, we will suppose that there exists a finite set of possible locations for the new facilities (discrete location space), and that customers follow a partially binary choice rule to serve their demand. So, the entering firm will serve a part of customers demand proportionally to the maximum attraction between each customer and the new facilities. It may happen that there are several new facilities with the maximum attraction for a customer, so it must decide if its demand is served by one or more of those facilities. But is this a real possible scenario? Can there be several facilities with the same maximum attraction for a customer? As the attraction depends on the characteristics of the facility (quality) and the distance to the customer, and it is feasible that the facilities belonging to the same firm have the same characteristics, the answer depends on whether it is usual for a customer to have several facilities of the same firm at the same distance. Although this situation is possible, this is unlikely to happen if real distances between facilities and consumers are considered. In fact, the customer does not use real distances to determine their attraction to a centre, but an approximation, which will depend on their accuracy when calculating distances between two points in a network, how often they use that route or the traffic in the area. In this way, a customer can consider that all facilities with the same characteristics whose real distances belong to an interval have the same attraction to him. This would justify that there may be several facilities with the same maximum attraction for a customer, being able to be equally chosen to serve their demand. To have a better estimation of the market share captured by each new facility, we suppose that in case of ties in the maximum attraction, customer's demand is equally split among all these facilities (equity-based rule, see Pelegrín *et al.*, 2015).

As location decisions are long-term, the firm wants not only to maximize its total market share, but also to ensure the viability of these facilities in the future, although the customers demand will be unknown and its preferences could change. In this way, the firm imposes the condition that a new facility will be opened only if it captures at least a fixed minimal demand in the short-term, trying to guarantee a future satisfactory captured demand. This will be introduced into the model as new constraints for candidate locations. So, it is necessary to know the exact captured demand by each new facility when locations must be decided. The equity-based ties breaking rule involves that both the objective function and the minimal market share constraints are nonlinear, so we will deal with a nonlinear constrained discrete competitive facility location model (see Balakrishnan and Storbeck, 1991; Benati and Hansen, 2002; Colomé *et al.*, 2003; Ljubić and Moreno, 2018; Serra *et al.*, 1999a for minimum capture constrained models and Drezner *et al.*, 2002; McGarvey and Cavalier, 2005; Plastria and Vanhaverbeke, 2008 for other constrained models).

This paper could be considered as an extension of Fernández *et al.* (2017), where unconstrained discrete competitive facility location models were analysed under binary and partially binary customer choice rules. There are some highlight differences between the previous paper and this new one that justify its relevance. In the previous one, none

ties breaking rule was used if ties in maximum attraction occur for the new facilities, since this was irrelevant for the entering firm when its market share was going to be maximized. In this new one, we deal with a nonlinear constrained problem due to the use of the equity-based ties breaking rule when there are several new facilities with maximal attraction for a customer. In addition, and for each customer, it has been considered that a new facility will capture part of the consumer's demand if the attraction between them exceeds a certain fixed threshold, so it may happen that there are customers with no captured demand by the new facilities.

In summary, the contribution of the paper is as follows. A new discrete location model is proposed where i) partially binary choice rule is applied to customers, ii) real distances are approximated by intervals of different ranges, iii) minimum attraction conditions have been considered for each customer, iv) an equity-based rule has been used in case of ties in maximum attraction between facilities, and v) minimal market share constraints are introduced to the new facilities unsure of a future captured demand.

The model is formulated as a binary nonlinear programming problem for which an equivalent formulation as a binary linear programming problem is proposed. This allows to solve small size problems using a standard optimizer (Xpress), and for more complex problems a heuristic algorithm based on ranking of the search space elements with constraints handling is proposed, which could also be used to solve other constrained problems. This algorithm extends the one proposed in Fernández *et al.* (2017).

The rest of the paper is organized as follows. In Section 2 the unconstrained model without ties breaking rule is described and an initial formulation is presented. In Section 3 the constrained model with equity-based ties breaking rule is introduced, the first nonlinear formulation is obtained, and then a linearization of the model is proposed. The proposed resolution method based on ranking of the search space elements is presented in Section 4, which could also be used to solve other constrained facility location problems. To justify the merit of the proposed algorithm, some computational experiments are shown in Section 5. Finally, conclusions are listed in the last section.

2. Partially Binary Basic Model

In this section we are going to present the basic model (unconstrained and without ties breaking rule), but introducing all the special characteristics of the model, and in the next section we will add additional constraints that ensure that a new facility will be opened only if it captures a minimal amount of customers demand. We will suppose that customers are concentrated in demand points, and that their demand is known and fixed. As it is very difficult for a customer to evaluate real distances to the facilities, we will consider distances by intervals, that is, taking integer values as the intervals centres, all distances within a fixed range interval will be approximated to its central value. Thus, for example, if we take a range of 10 km, all distances on the interval $(x - 5, x + 5]$ will be approximated to x , x being an integer value.

A new firm wants to compete for the market share by opening some new facilities in order to maximize the total captured demand. There are some pre-existing facilities

belonging to other firms, and the partially binary customer choice rule is considered. With this customer behaviour, the full demand of each customer is served by all the firms but patronizing only one facility from each one, the facility with the maximum attraction, then the demand is split between those facilities proportionally with their attraction. But there could be customers whose demand was not served by any of the new facilities because they were not attractive enough. To better adjust the problem to customer behaviour, we will assume that a customer will be served by a given facility if its attraction is greater than or equal to a fixed threshold value. For this, the attraction between each customer and its possible facilities will be defined as the ratio between the quality of the facility and a function of the distance between them, if this value is at least the threshold value set for the customer, and zero otherwise. For each customer, its threshold value will be between the minimum and the maximum attraction values of the most attractive facilities of the firms already operating in the market.

2.1. Notation

The following general notation is used:

Indices

- i, I index and set of demand points (customers),
- k, K index and set of pre-existing firms,
- j, J index and sets of facilities.

Data

- w_i demand at i ,
- q_j quality of facility j ,
- d_{ij} distance between demand point i and facility j considered by intervals of a fixed range,
- a_{ij} attraction that demand point i feels for facility j ,
- $a_i(J)$ maximum attraction that i feels for facilities in J ,
 $a_i(J) = \max \{a_{ij} : j \in J\}$,
- A_i minimum attraction required by customer i for a new facility,
- J_k pre-existing facilities of firm k in the market area,
- L set of possible locations for the new facilities.

Variables

- X set of new facilities locations, $X \subset L$, $|X| = s$.

Due to the minimum attraction condition of customers for the new facilities, for each $i \in I$ and $j \in L$:

$$a_{ij} = \begin{cases} \frac{q_j}{f(d_{ij})} & \text{if } \frac{q_j}{f(d_{ij})} \geq A_i, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

where $f(d_{ij})$ is an increasing function on distance d_{ij} .

If $M_{pb}(X)$ denotes the market share captured by the entering firm with facilities at X when partially binary customer choice rule is considered,

$$M_{pb}(X) = \sum_{i \in I} w_i \frac{a_i(X)}{a_i(X) + \sum_{k \in K} a_i(J_k)}$$

and the problem is

$$\text{Max}\{M_{pb}(X) : |X| = s, X \subset L\}.$$

If the following variables are considered:

$$\begin{aligned} x_j &= \begin{cases} 1 & \text{if a new facility is located at } j, \\ 0 & \text{otherwise,} \end{cases} \quad j \in L, \\ y_{ij} &= \begin{cases} 1 & \text{if } i \text{ is allocated to } j, \\ 0 & \text{otherwise,} \end{cases} \quad i \in I, j \in L \end{aligned} \quad (2)$$

the basic model has the following formulation as a binary nonlinear programming problem:

$$(PBBM) \left\{ \begin{array}{l} \text{Max } \sum_{i \in I} w_i \frac{\sum_{j \in L} a_{ij} y_{ij}}{\sum_{j \in L} a_{ij} y_{ij} + \sum_{k \in K} a_i(J_k)} \\ \text{s.t. } y_{ij} \leq x_j, \quad \forall i \in I, \forall j \in L, \\ \sum_{j \in L} y_{ij} = 1, \quad \forall i \in I, \\ \sum_{j \in L} x_j = s, \\ x_j \in \{0, 1\}, \quad \forall j \in L, \\ y_{ij} \in \{0, 1\}, \quad \forall i \in I, \forall j \in L. \end{array} \right.$$

Note that since the objective function is increasing with respect to the numerator, for each $i \in I$, the maximum captured demand for the entering firm will be obtained when i is allocated to a new facility j with maximum attraction, that is, $a_{ij} = \max\{a_{ir} : x_r = 1, r \in L\}$. In addition, as $\sum_{j \in L} y_{ij} = 1$, each i will be allocated to only one new facility with maximum attraction, following the partially binary customer choice rule. Note that constraints $y_{ij} \in \{0, 1\}$ can be replaced by $y_{ij} \geq 0$. At optimality, y_{ij} may be positive only if $a_{ij} = \max\{a_{ir} : x_r = 1, r \in L\}$.

3. Partially Binary Constrained Model

In order to introduce the constrained model, suppose that the entering firm imposes the condition that a new facility will be opened only if it captures at least a fixed amount of customers demand in the short-term denoted by α . With this new constraint, it is necessary to know what demand points are allocated to each new facility, and what part of the demand is captured by the entering firm. Given $i \in I$, it is known what proportion of its

demand will be captured by the entering firm, the proportion corresponding to its maximum attraction (this could be zero), but it is possible that there exist several new facilities with non zero maximum attraction. Since the minimal market share constraints refer to the new facilities, not to the entering firm, it is very important to know exactly the captured demand by each new facility, and then it is crucial to know what breaking rule is going to be used in case of ties in the maximum attraction. In this paper, instead of selecting only one of them to serve the full demand, we assume a more realistic situation, so, if ties occur, the demand of each customer captured by the entering firm will be equally split among its open facilities with maximum attraction. Then, we consider the following new variables:

$$z_{ij} = \begin{cases} 1 & \text{if } a_{ij} = \max\{a_{ir} : x_r = 1, r \in L\}, \\ 0 & \text{otherwise,} \end{cases} \quad i \in I, j \in L. \quad (3)$$

So the objective function becomes:

$$\text{Max} \sum_{i \in I} w_i \frac{\frac{\sum_{j \in L} a_{ij} z_{ij}}{\sum_{j \in L} z_{ij}}}{\frac{\sum_{j \in L} a_{ij} z_{ij}}{\sum_{j \in L} z_{ij}} + \sum_{k \in K} a_i(J_k)} \quad (4)$$

$$= \sum_{i \in I} w_i \frac{\sum_{j \in L} a_{ij} z_{ij}}{\sum_{j \in L} a_{ij} z_{ij} + (\sum_{k \in K} a_i(J_k)) \sum_{j \in L} z_{ij}}, \quad (5)$$

where it must be verified that:

$$z_{ij} = 1 \quad \Leftrightarrow \quad a_{ij} = \max\{a_{ir} : x_r = 1, r \in L\}. \quad (6)$$

The necessary condition in (6) is verified by any optimal solution, since the objective function is increasing with respect to the numerator. To guarantee the sufficient condition in (6), besides that a demand point can only be allocated to open facilities, the following set of constraints is necessary:

$$\epsilon z_{ij} + \left(\sum_{h \in L} z_{ih} a_{ih} - \left(\sum_{h \in L} z_{ih} \right) a_{ij} \right) + |L|(1 - x_j) a_{\max} \geq \epsilon, \quad \forall i \in I, \forall j \in L, \quad (7)$$

where ϵ is small enough and $a_{\max} = \text{Max}\{a_{ij} : \forall i \in I, \forall j \in L\}$.

Note that if $a_{ij} = \max\{a_{ir} : x_r = 1, r \in L\}$ and $z_{ij} = 0$, then the left side of (7) becomes $\sum_{h \in L} z_{ih} a_{ih} - (\sum_{h \in L} z_{ih}) a_{ij} \leq 0$ and then (7) doesn't hold. On the other hand, if $a_{ij} < \max\{a_{ir} : x_r = 1, r \in L\} = a_{ik}$ and $z_{ij} = 1$, then for $j = k$ the left side of (7) becomes $\sum_{h \in L} z_{ih} a_{ih} - (\sum_{h \in L} z_{ih}) a_{ij} < 0$ and (7) doesn't hold.

Moreover, from (7) it follows that $\sum_{j \in L} z_{ij} \geq 1, \forall i \in I$, even if $\max\{a_{ir} : x_r = 1, r \in L\} = 0$, so the objective function is well defined.

On the other hand, the minimal market share constraint for each new facility, when the previously introduced ties breaking rule is considered, is expressed as:

$$\sum_{i \in I} w_i \frac{\frac{a_{ij} z_{ij}}{\sum_{j \in L} z_{ij}}}{\frac{\sum_{j \in L} a_{ij} z_{ij}}{\sum_{j \in L} z_{ij}} + \sum_{k \in K} a_i(J_k)} \geq \alpha x_j, \quad \forall j \in L \quad \Leftrightarrow \quad (8)$$

$$\sum_{i \in I} w_i \frac{a_{ij} z_{ij}}{\sum_{j \in L} a_{ij} z_{ij} + (\sum_{k \in K} a_i(J_k)) \sum_{j \in L} z_{ij}} \geq \alpha x_j, \quad \forall j \in L \quad (9)$$

that is a similar expression than the objective function, so it is also well defined.

So, the final formulation of the model as a binary nonlinear programming problem is:

$$(PBCM_{\alpha}) \left\{ \begin{array}{l} \text{Max } \sum_{i \in I} w_i \frac{\sum_{j \in L} a_{ij} z_{ij}}{\sum_{j \in L} a_{ij} z_{ij} + (\sum_{k \in K} a_i(J_k)) \sum_{j \in L} z_{ij}} \\ \text{s.t. } z_{ij} \leq x_j, \quad \forall i \in I, \quad \forall j \in L, \\ \sum_{j \in L} x_j = s, \\ \sum_{i \in I} w_i \frac{a_{ij} z_{ij}}{\sum_{j \in L} a_{ij} z_{ij} + (\sum_{k \in K} a_i(J_k)) \sum_{j \in L} z_{ij}} \geq \alpha x_j, \quad \forall j \in L, \\ \epsilon z_{ij} + (\sum_{h \in L} z_{ih} a_{ih} - (\sum_{h \in L} z_{ih}) a_{ij}) + |L|(1 - x_j) a_{\max} \geq \epsilon, \\ \quad \forall i \in I, \quad \forall j \in L, \\ x_j \in \{0, 1\}, \quad \forall j \in L, \\ z_{ij} \in \{0, 1\}, \quad \forall i \in I, \quad \forall j \in L. \end{array} \right.$$

3.1. Linearization of the Constrained Model

(PBCM_α) model is a nonlinear programming problem due to both the objective function and the minimal market share constraints set, but both can be linearized using the following sets of variables (see Benati and Hansen, 2002; Biesinger et al., 2016; Fernández et al., 2017; Haase and Muller, 2014; Kochetov et al., 2013 for similar linearizations):

$$q_i = \frac{1}{\sum_{j \in L} a_{ij} z_{ij} + (\sum_{k \in K} a_i(J_k)) \sum_{j \in L} z_{ij}}, \quad \forall i \in I, \quad (10)$$

$$v_{ij} = q_i z_{ij}, \quad \forall i \in I, \quad \forall j \in L,$$

where $0 \leq q_i \leq M$, being $M = \max_i \frac{1}{\sum_k a_i(J_k)}$.

Then, the discrete competitive facility location model with partially binary customer choice rule and minimal market share constraints (PBCM_α) has the following formulation as a mixed binary linear programming problem:

$$\left. \begin{array}{l}
 \text{Max } \sum_{i \in I} \sum_{j \in L} w_i a_{ij} v_{ij} \\
 \text{s.t. } \sum_{j \in L} a_{ij} v_{ij} + (\sum_{k \in K} a_i(J_k)) \sum_{j \in L} v_{ij} = 1, \forall i \in I, \\
 v_{ij} \leq M z_{ij}, \forall i \in I, \forall j \in L, \\
 v_{ij} \leq q_i, \forall i \in I, \forall j \in L, \\
 q_i \leq v_{ij} + M(1 - z_{ij}), \forall i \in I, \forall j \in L, \\
 z_{ij} \leq x_j, \forall i \in I, \forall j \in L, \\
 \sum_{j \in L} x_j = s, \\
 \sum_{i \in I} w_i a_{ij} v_{ij} \geq \alpha x_j, \forall j \in L, \\
 \epsilon z_{ij} + \sum_{r \in L} a_{ir} z_{ir} - (\sum_{r \in L} z_{ir}) a_{ij} + |L|(1 - x_j) a_{max} \geq \epsilon, \\
 \forall i \in I, \forall h \in L, \\
 x_j \in \{0, 1\}, \forall j \in L, \\
 q_i \geq 0, \forall i \in I, \\
 z_{ij} \in \{0, 1\}, v_{ij} \geq 0, \forall i \in I, \forall j \in L.
 \end{array} \right\} \text{(LPBCM}_\alpha\text{)}$$

4. Ranking-Based Random Search

The Ranking-based Discrete Optimization Algorithm (RDOA) has been applied to solve facility location problems in Fernández *et al.* (2017). The RDOA is specially adopted to discrete facility location and is based on ranking of candidate locations. The algorithm demonstrated good efficiency when solving different instances of unconstrained discrete facility location problems and outperformed well-known algorithms suitable to solve similar problems.

However, the RDOA does not have any strategy for constraint handling and, therefore, cannot be directly applied to solve our constrained facility location model. In this section we propose a modified RDOA, which includes constraints for the solutions (RCDOA).

4.1. Ranking-Based Constrained Discrete Optimization Algorithm

Ranking-based Constrained Discrete Optimization Algorithm (RCDOA) starts with an initial solution

$$Y = \{y_1, y_2, \dots, y_s\}, \quad (11)$$

which is a subset of all candidate locations L and is considered as the best solution found so far. Here s is the number of facilities expected to locate. The initial solution can be generated at random or obtained by other optimization methods, but must be feasible according to the problem constraints.

A new solution

$$Y' = \{y'_1, y'_2, \dots, y'_s\} \quad (12)$$

is generated by in turn taking candidate locations from the best known solution Y and changing them to another ones randomly selected from the set of all possible candidate locations excluding those which already form Y or Y' . Each location y_k is changed to probability $1/s$, and is copied without changing to probability $1 - 1/s$:

$$y'_k = \begin{cases} l \in L \setminus (Y \cup Y'), & \text{if } \xi_k < 1/s, \\ y_k, & \text{otherwise,} \end{cases} \quad (13)$$

where ξ_k is a random number uniformly generated over the interval $[0, 1]$, and $i = 1, 2, \dots, s$. Location l is randomly selected among the location candidates not already included in $Y \cup Y'$ using a roulette wheel strategy.

A candidate location $l_i \in L$ can be selected with probability

$$\pi_i = \frac{r_i}{d(l_i, y_k) \sum_{j=1}^{|L|} \frac{r_j}{d(l_j, y_k)}}, \quad (14)$$

where r_i is a rank of candidate location l_i and $d(l_i, y_k)$ is a geographical distance between candidate location $l_i \in L$ and candidate location $y_k \in Y$ which is being changed ($k = 1, 2, \dots, s$).

The ranks of all candidate locations initially are equal to 1 and are automatically adjusted depending on success and failures when generating a new solution. If the newly generated solution Y' is feasible and market share $M(Y')$ is greater than the market share $M(Y)$ of the best known solution, then (1) the ranks of all locations which form better solution Y' are increased by one and (2) the ranks of all locations that form outperformed solution Y , but do not form Y' , are reduced by one:

$$r_i = \begin{cases} r_i + 1, & \text{if } l_i \in Y', \\ r_i - 1, & \text{if } l_i \in Y \setminus Y', \\ r_i, & \text{otherwise.} \end{cases} \quad (15)$$

If the newly generated solution Y' is not feasible or $M(Y')$ is not greater than $M(Y)$, then the ranks of all candidate locations forming unsuccessfully generated solution Y' , but do not forming the best known solution Y , are reduced by one:

$$r_i = \begin{cases} r_i - 1, & \text{if } l_i \in Y' \setminus Y, \\ r_i, & \text{otherwise.} \end{cases} \quad (16)$$

If the newly generated solution outperforms the best solution found so far, then Y is changed to Y' and iteration is assumed to be successful; otherwise, Y remains unchanged and iteration is assumed to be unsuccessful.

4.2. Pre-Optimization Stage

In order to take advantage of the RCDOA, the best known solution Y used to generate new ones must be feasible. However, finding a feasible solution can be a non-trivial task, requiring significant computational effort.

The pre-optimization stage is used to find a feasible initial solution for RCDOA, which starts with a randomly generated solution, which can even be infeasible. If the generated solution is feasible, then the pre-optimization stage is skipped and the optimization procedure continues with the RCDOA.

The constraint for minimal market share obliges to find locations for the new facilities so that each facility would attract at least a given part α of the market share. Otherwise the solution is treated as infeasible and cannot be considered as a candidate solution for the problem. Let's denote by m_1, m_2, \dots, m_s the market share captured by facilities located in $y_i \in Y$. Then violation of the minimal market share constraint of facility located in y_i is expressed by market share deficiency:

$$v_i = \begin{cases} \alpha - m_i, & \text{if } m_i < \alpha, \\ 0, & \text{otherwise,} \end{cases} \quad (17)$$

where $i = 1, 2, \dots, s$. The violation of the whole solution Y is

$$V(Y) = \sum_{i=1}^s v_i. \quad (18)$$

Equation (17) guarantees that (18) is a sum of non-negative values. Thus makes $V(Y)$ a non-negative function with zero lower bound, which indicates that solution Y is feasible; larger $V(Y)$ indicates larger violation of the constraint.

The pre-optimization stage is based on solution of unconstrained optimization subproblem with objective function $V(Y)$, which is subject to minimize:

$$Y^* = \arg \min_{Y \subset L} V(Y). \quad (19)$$

A new solution Y' is generated by changing the best known solution Y using the same strategy as in RDOA – the algorithm for unconstrained optimization. The ranking of candidate locations is also performed using similar strategy as in RDOA: if $V(Y')$ is lower than $V(Y)$, then ranks of the candidate locations forming Y' are increased by one and ranks of the candidate locations in $Y \setminus Y'$ are reduced by one; similarly, if $V(Y')$ is not lower than $V(Y)$, then ranks of the candidate locations in $Y' \setminus Y$ are reduced by one.

The stopping criterion of the pre-optimization stage is formulated as to find the lower bound of the objective function value. If a decision variable Y such that $V(Y) = 0$ cannot be found using a predefined number of function evaluations or within a reasonable time devoted to solve the whole optimization problem, then it is stated that a feasible solution cannot be found. The complexity of the pre-optimization stage, depends on the optimization problem instance and will be investigated hereinafter.

After a feasible solution is found, the algorithm proceeds to the next stage aimed at maximization of the market share for the new facilities using RCDOA with a feasible initial solution found in pre-optimization stage. The RCDOA starts with ranks of candidate locations adjusted when looking for feasible solutions: candidate locations that fail in forming a feasible solution have lower ranks comparing to other candidate locations.

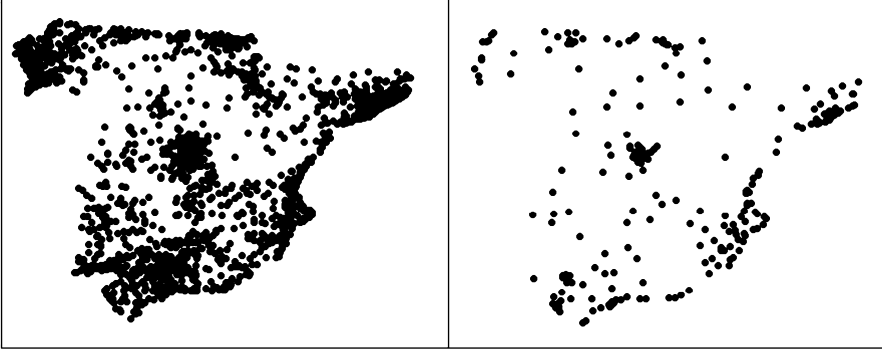


Fig. 1. Left: demand points. Right: location candidates.

Table 1
Distribution of preexisting locations.

Firm	Indices of demand points	
	3 facilities per firm	5 facilities per firm
J_1	1, 4, 7	1, 4, 7, 10, 13
J_2	2, 5, 8	2, 5, 8, 11, 14
J_3	3, 6, 9	3, 6, 9, 12, 15

5. Experimental Investigation

The (LPBCM $_{\alpha}$) described in Section 3 has been solved by the deterministic optimization algorithm using Xpress (FICO Xpress Mosel, 2014) and the RCDOA, described in Section 4.

The database with real geographical data of coordinates of 1624 municipalities in Spain with population greater or equal to 2500 inhabitants has been considered as demand points. The set of 247 most populated demand points (municipalities with a population greater or equal to 25000 inhabitants) was used as the set of location candidates L (see Fig. 1).

For distances between demand points and facilities, two different ranges have been used on computational experiments, 10 and 20 km, and for the attraction function, it has been supposed that $q_j = 1, \forall j \in L$, and $f(d_{ij}) = 1 + d_{ij}, \forall i \in I, j \in L$.

It was considered that there are 3 preexisting firms with 3 or 5 facilities per firm already located in the most populated demand points. Distribution of candidate locations among the firms is presented in Table 1.

The minimal market share for a new facility is expressed as percent of estimated average market share per existing facility after location. The average market share per facility can be expressed by

$$\tilde{m} = \frac{W}{|J_1| + |J_2| + |J_3| + s}, \quad (20)$$

Table 2
Minimal market share per new facility (α) for different (PBCM $_{\alpha}$) instances.

(J_k , s)	0%	10%	20%	30%	40%	50%	60%	70%	80%
(3, 5)	0	253310	506620	759930	1013240	1266550	1519860	1773170	2026480
(3, 10)	0	186650	373300	559950	746600	933250	1119900	1306550	1493200
(5, 5)	0	177310	354620	531930	709240	886550	1063860	1241170	1418480
(5, 10)	0	141850	283700	425550	567400	709250	851100	992950	1134800

where

$$W = \sum_{i=1}^{|I|} w_i \quad (21)$$

is the total demand of all customers in I . Then the minimal market share is considered as 0% to 80% of \tilde{m} , depending on the problem instance, but only the results for some of them will be shown, the ones with significative information for the model. For small α values, Xpress is able to find the optimal solutions, which do not usually change, and only when α values increase, the optimal solutions can change, if any, and it is even possible that Xpress can not find them because of the time limit. The values of minimal market share α are presented in Table 2, knowing that the total demand W of I is 35462869.

To define A_i , the minimum attraction level for customer i , we consider $A_i^{\max} = \max\{a_i(J_k) : k \in K\}$ and $A_i^{\min} = \min\{a_i(J_k) : k \in K\}$, then $A_i = A_i^{\min} + \delta(A_i^{\max} - A_i^{\min})$ with $0 \leq \delta \leq 1$. Four δ values have been selected for the result tables, $\delta = 0, 0.2, 0.6, 1$, in order to reduce the table size.

5.1. Validation of RCDOA

The set of 144 instances for each distance range with different combinations of the number of pre-existing facilities per firm ($|J_k|$), the number of facilities expected to locate (s), the minimal market share α and δ values were solved using Xpress and RCDOA. The computations with Xpress have been executed until the whole search space was explored or the predefined time limit was exceeded (18000 s). The RCDOA has been executed for 10000 function evaluations. Due to stochastic nature of RCDOA, each problem instance was solved 100 times to get statistically significant results.

The results, obtained solving (LPBCM $_{\alpha}$) instances with 3 preexisting facilities per firm, are presented on the left part of Tables 3 and 4, and results, obtained solving (LPBCM $_{\alpha}$) instances with 5 preexisting facilities per firm, on the right part of the tables, but only for α (%) = 0, 30, 60, 70 and 80, since only for large α values Xpress has problems to obtain the optimal solutions. The acronym UNF in the table means that Xpress was unable to complete computation within the predefined time limit (unfinished). The acronym FNF stands for Feasible Not Found and means that an algorithm was unable to find a feasible solution – within a predefined time with Xpress and within 10000 function evaluations with RCDOA.

Table 3
Results obtained using Xpress and RCDOA for solving (LPBCM $_{\alpha}$) with 10 km as distance range.

(J_k , s)	δ	α (%)	$M(Y)$		(J_k , s)	δ	α (%)	$M(Y)$	
			XPress	RDOA				XPress	RDOA
(3, 5)	0	0	12819396	12819396	(5, 5)	0	0	10936974	10936974
		30	12819396	12819396			30	10936974	10936974
		60	12819396	12819396			60	10936974	10936974
		70	12819396	12819396			70	10936974	10936974
		80	UNF	FNF			80	UNF	FNF
	0.2	0	12342550	12342550	0.2	0	10150267	10150267	
		30	12342550	12342550		30	10150267	10150267	
		60	12342550	12342550		60	10150267	10150267	
		70	UNF	FNF		70	10026202	10026202	
		80	UNF	FNF		80	UNF	FNF	
	0.6	0	11579618	11579618	0.6	0	9552229	9552229	
		30	11579618	11579618		30	9552229	9552229	
		60	11579618	11579618		60	9552229	9552229	
		70	UNF	FNF		70	8679257	8679257	
		80	FNF	FNF		80	FNF	FNF	
1	0	11177870	11177870	1	0	9040555	9040555		
	30	11177870	11177870		30	9040555	9040555		
	60	UNF	FNF		60	8435659	8435659		
	70	FNF	FNF		70	FNF	FNF		
	80	FNF	FNF		80	FNF	FNF		
(3, 10)	0	0	16128566	16128566	(5, 10)	0	0	14110653	14110653
		30	16128566	16128566			30	14110653	14110653
		60	UNF	FNF			60	UNF	FNF
		70	UNF	FNF			70	UNF	FNF
		80	UNF	FNF			80	UNF	FNF
	0.2	0	16122297	16122297	0.2	0	13969654	13969654	
		30	16122297	16122297		30	13969654	13969654	
		60	UNF	FNF		60	UNF	FNF	
		70	UNF	FNF		70	13595768	13595768	
		80	FNF	FNF		80	UNF	FNF	
	0.6	0	15995766	15995766	0.6	0	13429857	13429857	
		30	15995766	15995766		30	13429857	13429857	
		60	15784480	15784480		60	UNF	FNF	
		70	UNF	FNF		70	UNF	FNF	
		80	FNF	FNF		80	FNF	FNF	
1	0	15887608	15887608	1	0	12978556	12978556		
	30	15887608	15887608		30	12978556	12978556		
	60	15489291	15489291		60	11925893	11925893		
	70	FNF	FNF		70	FNF	FNF		
	80	FNF	FNF		80	FNF	FNF		

The computations with Xpress last from 30 seconds (the fastest) to around 14000 seconds (when optimal solutions found), or remain unfinished. The CPU times are similar for both distance ranges, perhaps somewhat higher for the 20 km range when δ values increase. In general, times increase when α or δ increase and the rest of the parameters are fixed. When δ increases, the minimum level of attraction for customers increases, and

Table 4
 Results obtained using Xpress and RCDOA for solving (LPBCM $_{\alpha}$) with 20 km as distance range.

(J_k , s)	δ	α (%)	$M(Y)$		(J_k , s)	δ	α (%)	$M(Y)$	
			XPress	RDOA				XPress	RDOA
(3, 5)	0	0	12862282	12862282	(5, 5)	0	0	11043517	11043517
		30	12862282	12862282			30	11043517	11043517
		60	12862282	12862282			60	11043517	11043517
		70	12862282	12862282			70	11043517	11043517
		80	UNF	FNF			80	UNF	FNF
	0.2	0	12406408	12406408		0.2	0	10356344	10356344
		30	12406408	12406408			30	10356344	10356344
		60	12406408	12406408			60	10356344	10356344
		70	UNF	FNF			70	10254115	10254115
		80	UNF	FNF			80	UNF	FNF
	0.6	0	11645211	11645211		0.6	0	9746979	9746979
		30	11645211	11645211			30	9746979	9746979
		60	11645211	11645211			60	9746979	9746979
		70	UNF	FNF			70	9018939	9018939
		80	FNF	FNF			80	FNF	FNF
1	0	11212067	11212067	1	0	9272288	9272288		
	30	11212067	11212067		30	9272288	9272288		
	60	FNF	FNF		60	9272288	9272288		
	70	FNF	FNF		70	7936323	7936323		
	80	FNF	FNF		80	FNF	FNF		
(3, 10)	0	0	16266416	16266416	(5, 10)	0	0	14291976	14291976
		30	16266416	16266416			30	14291976	14291976
		60	UNF	FNF			60	UNF	FNF
		70	UNF	FNF			70	14210030	14210030
		80	UNF	FNF			80	UNF	FNF
	0.2	0	16224017	16224017		0.2	0	14185476	14185476
		30	16224017	16224017			30	14185476	14185476
		60	UNF	FNF			60	14050852	14050852
		70	UNF	FNF			70	14050852	14050852
		80	UNF	FNF			80	FNF	FNF
	0.6	0	16040410	16040410		0.6	0	13643691	13643691
		30	16040410	16040410			30	13643691	13643691
		60	15950660	15950660			60	13310686	13310686
		70	UNF	FNF			70	UNF	FNF
		80	FNF	FNF			80	FNF	FNF
1	0	15938689	15938689	1	0	13259531	13259531		
	30	15938689	15938689		30	13259531	13259531		
	60	15668471	15668471		60	FNF	FNF		
	70	FNF	FNF		70	FNF	FNF		
	80	FNF	FNF		80	FNF	FNF		

it is possible that instances that were feasible become infeasible because of the decreasing in the number of demand points that each facility can capture. If α and/or δ increase too much, the instances can be infeasible. Meanwhile, RCDOA performs its 10000 function evaluations within less than a half minute independent of the instance.

Table 5
Results obtained using RCDOA for solving CFLP with 1000 location candidates and 10 km as distance range.

(J_k , s)	δ	α	$M(Y)$			FNF	FFA	
			Avg	Min	Max			
(5, 5)	0	0	10943317	10936974	10950470	0	1	
		30	10942777	10936974	10950470	0	3	
		60	10941666	10758392	10950470	0	61	
		70	10940315	10744730	10950470	0	321	
		80	7379300	7150127	9862744	96	580	
	0.2	0	10146666	9995868	10167048	0	1	
		30	10146520	10008293	10167048	0	4	
		60	10143395	9978345	10167048	0	311	
		70	9932472	7879719	10044660	43	1939	
		80	N/A	N/A	N/A	100	N/A	
		0	9526613	9259547	9552229	0	1	
		30	9535052	9410487	9552229	0	15	
		60	9450394	5837770	9552229	2	1716	
	0.6	70	8637104	6888141	8726454	67	2723	
		80	N/A	N/A	N/A	100	N/A	
		(5, 10)	0	0	14076637	13936239	14128881	0
30				14071104	13930319	14128881	0	12
60	13454097			8845844	13751062	3	883	
70	12915529			10160219	13751062	19	1685	
80	N/A			N/A	N/A	100	N/A	
0.2	0		13910487	13797369	13987882	0	1	
	30		13918547	13702294	13987882	0	16	
	60		12801503	9333460	13607286	2	860	
	70		13229673	12115228	13605241	77	2518	
	80		N/A	N/A	N/A	100	N/A	
0.6	0		13393572	13265791	13448805	0	1	
	30		13393438	13180237	13448805	0	33	
	60		12264524	12168197	12316132	93	3603	
	70		N/A	N/A	N/A	100	N/A	
	80		N/A	N/A	N/A	100	N/A	

All optimal solutions found by Xpress were also identified by RCDOA, and the results for the selected δ and α values can be seen on Tables 3 and 4. Although, solutions found by RCDOA only are not proven to be optimal, their objective values are similar to the optimal ones found by Xpress when solving similar problem instances.

5.2. Investigation of RCDOA

The RCDOA was applied to solve the CFLP with the same set of demand points, but with 1000 location candidates and 10 km distance range for measuring attractiveness of facilities. The budget of 10000 function evaluation was fixed for solution of a single instance. Due to the stochastic nature of the algorithm each problem was solved 100 times (starting from a randomly generated initial solution) and statistical results were recorded.

The results are presented in Table 5, where average, minimal, and maximal objective function values of the best solution are presented. Additionally, the number of trials when

a feasible solution was not found (FNF) and average number of function evaluations required to find a feasible solution (FFA, Feasible Found After) are presented in the table. If the algorithm fails to find a feasible solution for some trials, these trials are ignored and statistics is derived from trials in which a feasible solution was found. The acronym N/A means that statistics are not available because the algorithm fails to find a feasible solution in all trials.

One can see from Table 5 that RCDOA is able to find a feasible solution for most of problem instances. A feasible solution can be found within 1000 function evaluations for all instances with $\alpha \leq 30\%$. The instances with large α values required more than 1000 function evaluations to determine a feasible solution. A feasible solution was not found for instances with larger constraint for the minimal market share ($\alpha = 80\%$ in most cases and $\alpha = 70\%$ for one instance), except the instance with $|J_k| = 5$, $s = 5$, and $\delta = 0$ where a feasible solution was found even using the largest constraint.

The best solutions found for the CFLP with 1000 location candidates give larger objective function values than the optimal solutions for the same problem instances but with smaller set of location candidates (see Table 3). Moreover, a feasible solution was found for some instances, for which Xpress and RCDOA failed when the set of location candidates was smaller. This is because a larger set of location candidates lets us form better solutions and RCDOA is able to determine them.

Table 6 shows probability to find the best known solution and its approximation with up to 5% error. One can see from the table that probability depends on the values of δ and α : larger values – smaller probability. On the other hand, it is probability equal to 1 to find approximation with 5% error for most problem instances, except for 7 instances, where probability was lower, and instances where feasible solution was not found at all.

6. Conclusions

In this paper a new constrained discrete competitive facility location model has been analysed in which, besides assuming that customers follow a partially binary choice rule for selecting the facility or facilities that will serve their demand, equity-based ties breaking rule is proposed and minimal market share constraints have been incorporated to ensure the viability of the new facilities (a new facility will be opened if it captures a minimal amount of customer demand in that area). A formulation as a nonlinear binary programming problem has been proposed, and after defining new sets of variables, it has been possible to present a linearization of the proposed constrained model (LPBCM $_{\alpha}$), which allows to obtain optimal solutions for small size problems using a standard optimizer.

Due to the complexity of the (LPBCM $_{\alpha}$) model, a heuristic algorithm for random search based on ranking of location candidates and geographical distance has been applied for the resolution of more complex problems. This procedure uses two labels for each location candidate point: the ranking label and the violation label of the minimal market share constraint. The viability of the proposed algorithm has been investigated for different problem instances and compared with the standard optimizer Xpress, showing that RCDOA always obtains the same solutions as Xpress. For more complex instances,

Table 6
Probability to find the optimal solution and its approximation with up to 5% error using RCDOA.

(J_k , s)	δ	α	Probability to find the best known solution						
			0%	1%	2%	3%	4%	5%	
(5, 5)	0	0	0.47	1.00	1.00	1.00	1.00	1.00	1.00
		30	0.43	1.00	1.00	1.00	1.00	1.00	1.00
		60	0.48	0.99	1.00	1.00	1.00	1.00	1.00
		70	0.39	0.99	1.00	1.00	1.00	1.00	1.00
		80	0.25	0.75	1.00	1.00	1.00	1.00	1.00
	0.2	0	0.15	0.98	1.00	1.00	1.00	1.00	1.00
		30	0.12	0.97	1.00	1.00	1.00	1.00	1.00
		60	0.12	0.96	1.00	1.00	1.00	1.00	1.00
		70	0.40	0.89	0.89	0.89	0.90	0.89	0.89
		80	N/A	N/A	N/A	N/A	N/A	N/A	N/A
	0.6	0	0.14	0.96	0.97	0.97	1.00	1.00	1.00
		30	0.18	0.99	1.00	1.00	1.00	1.00	1.00
		60	0.19	0.98	0.98	0.98	1.00	0.98	0.98
		70	0.21	0.97	0.97	0.97	1.00	0.97	0.97
		80	N/A	N/A	N/A	N/A	N/A	N/A	N/A
(5, 10)	0	0	0.06	0.97	1.00	1.00	1.00	1.00	1.00
		30	0.06	0.93	1.00	1.00	1.00	1.00	1.00
		60	0.04	0.76	0.89	0.91	0.90	0.95	0.95
		70	0.01	0.69	0.72	0.73	0.80	0.88	0.88
		80	N/A	N/A	N/A	N/A	N/A	N/A	N/A
	0.2	0	0.01	0.88	1.00	1.00	1.00	1.00	1.00
		30	0.02	0.92	0.99	1.00	1.00	1.00	1.00
		60	0.01	0.54	0.64	0.65	0.70	0.66	0.66
		70	0.04	0.52	0.61	0.61	0.60	0.61	0.61
		80	N/A	N/A	N/A	N/A	N/A	N/A	N/A
	0.6	0	0.06	0.89	1.00	1.00	1.00	1.00	1.00
		30	0.01	0.89	1.00	1.00	1.00	1.00	1.00
		60	0.14	0.86	1.00	1.00	1.00	1.00	1.00
		70	N/A	N/A	N/A	N/A	N/A	N/A	N/A
		80	N/A	N/A	N/A	N/A	N/A	N/A	N/A

only RCDOA has been used, and the results show that in all cases, the algorithm either obtains the best possible solution, or converges to its approximation if a feasible solution is available.

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