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Neighbourhood and Lattice Models of Second-Order Intuitionistic Propositional Logic

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Abstract. We study a version of the Stone duality between the Alexandrov spaces and the completely distributive algebraic lattices. This enables us to present lattice-theoretical models of second-order intuitionistic propositional logic which correlates with the Kripke models introduced by Sobolev. This can be regarded as a second-order extension of the well-known correspondence between Heyting algebras and Kripke models in the semantics of intuitionistic propositional logic.

Keywords: second-order intuitionistic propositional logic, complete Heyting algebra, completeness theorem

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1. Introduction

Taking into account semantic aspects of intuitionistic propositional logic, we can find at least two types of model presentation [1, 2, 3], both of which enable us to show soundness and completeness of the formal system. One is based on a topological viewpoint known as Kripke models and the other is based on a lattice-theoretical viewpoint known as Heyting algebras. These two frameworks seem to be equivalent in the sense that there exists a two-way translation between them which preserves the validity of every proposition. Indeed, from a Kripke model, we can construct the Heyting algebra by the set of upward-closed sets of possible worlds. On the other hand, from a Heyting algebra, we can generate all possible worlds of the Kripke model by considering prime filters on the Heyting algebra.

In contrast, directing our attention to the formal system $\mathbf{NJ_2}$ of the second-order intuitionistic propositional logic, it is quite opaque whether this neat correspondence is still established or not. Indeed, on one hand, we can find a notion of Kripke models introduced by Sobolev [4], for which $\mathbf{NJ_2}$ is shown to be sound and complete. However, on the other hand, we can never find any successful framework as its lattice-theoretical counterpart.

If we intend to interpret a proposition of the form $\forall p.A$ by a lattice-theoretical framework, then it would be natural to demand a structure of a complete lattice and to define the denotation of $\forall p.A$ as the greatest lower bound of all denotations of A where the valuation of p ranges over all elements of the complete lattice. However, this way of interpretation involves a difficulty in demonstrating the completeness theorem, as is pointed out in [5]. To overcome it, one reasonable approach would be to weaken a requirement for the underlying structure of complete lattice so as to regard the Lindenbaum algebra induced from the propositions of $\mathbf{NJ_2}$ as a model. In fact, this kind of generalization has been successful in various standpoints of semantics [6], and Kremer [7] actually gives a framework of models for which the completeness of $\mathbf{NJ_2}$ is ensured. In contrast with these positive aspects, any structures appearing in these discussions would not be comparable with Sobolev's framework because in his definition the range of the valuation of bound variables sensitively depends on the possible world in which we have to define the interpretation.

In this context, we present a framework of lattice models which correlates strictly with Sobolev's framework, for which $\mathbf{NJ_2}$ is necessarily shown to be sound and complete. This is accomplished by representing Sobolev's construction in terms of complete Heyting algebras through a certain topological viewpoint. We notice that Sobolev's discussion deeply depends on the device of Alexandrov topology, and it can be generalized to a version of neighbourhood semantics [8, 9] which has been extensively studied in the semantics of modal logic. Furthermore, we can discuss a neat correspondence between topological spaces and complete Heyting algebras, which underlies the construction of our lattice models. It would be natural to make use of the correspondence known as the Stone duality, that is, the dual equivalence between various categories of sober spaces and those of spatial lattices. Actually, this enables us to introduce a definition of models of $\mathbf{NJ_2}$ based on spatial lattices, which is mainly discussed in [10]. However, the models so obtained are not sufficient to ensure the completeness because Sobolev's canonical model is excluded by the fact that Alexandrov spaces are not always sober.

In spite of the difficulty above, we establish a version of the Stone duality between the category $\mathbf{Alex}$ of Alexandrov spaces and the category $\mathbf{CDA}$ of completely distributive algebraic lattices, which naturally leads us to an issue of lattice models for $\mathbf{NJ_2}$. As a matter of fact, it enables us to denote every proposition of $\mathbf{NJ_2}$ as a set of compact, completely prime filters on a complete Heyting algebra while preserving the validity based on Sobolev's semantics. This justifies the soundness and completeness of $\mathbf{NJ_2}$ with respect to a lattice-theoretical viewpoint.

The content of this paper proceeds as follows. In Section 3, we first review the presentation of the completeness theorem by Sobolev. Then we generalize it from a topological viewpoint and introduce a framework of neighbourhood models. Although Sobolev's semantics is a special case of our neighbourhood semantics, the soundness of $\mathbf{NJ_2}$ is still ensured with respect to our generalized standpoint. In Section 4, we focus on the category $\mathbf{Alg}$ of algebraic domains equipped with Scott topology for mediating between $\mathbf{Alex}$ and $\mathbf{CDA}$. Actually, the operation of ideal completion gives a translation from $\mathbf{Alex}$ and $\mathbf{Alg}$ and its inverse is presented as a relativization of Scott topology. So this correspondence and the well-known Stone duality between $\mathbf{Alg}$ and $\mathbf{CDA}$ turn out to give a dual equivalence between $\mathbf{Alex}$ and $\mathbf{CDA}$, which satisfies all requirements for our model construction. In Section 5, we present the definition of our lattice models, for which we show soundness and completeness of $\mathbf{NJ_2}$. To show this, we translate the structure of neighbourhood models in terms of complete Heyting algebras according to the dual equivalence studied in Section 4, through which the information of satisfiability is preserved.

2. Basic notions

The above-mentioned discussion depends on the theories of categories, topological spaces and partially ordered sets. Here we briefly review some fundamental terminologies and notations. For more details about them, see [11, 12, 13]. In the discussion below, denoting a topological space, we often omit to indicate the topology if there is no possibility of confusion. Similarly, denoting a poset, we often omit to indicate the order relation.

Let $\langle L, \sqsubseteq \rangle$ be a poset, and P a subset of L. We say that P is directed if every finite subset of P has an upper bound in P. We also say that P is an ideal on L if it is directed and downward closed. For every $a \in L$, it is clear that the set $\downarrow a = \{b \in L \mid b \sqsubseteq a\}$ is an ideal, which is called the principal ideal generated from a. We define filters and principal filters as the dual notions of the ideals and the principal ideals, respectively. The poset $\langle L, \sqsubseteq \rangle$ is said to be complete if every directed subset of L has its least upper bound in L, and we specifically designate the least upper bound of P by $\bigsqcup^{\uparrow} P$ in case where P is directed[1].

It is known that every complete Heyting algebra is characterized as a complete lattice $\langle L, \sqsubseteq \rangle$ which satisfies the frame distributivity law, that is, $a \sqcap (\bigsqcup P) = \bigsqcup\{a \sqcap b \mid b \in P\}$ for every $a \in L$ and $P \subseteq L$. We let $\mathbf{cHey}$ stand for the category of complete Heyting algebras. More precisely, the class of its objects consists of all complete Heyting algebras, and the class of its arrows consists of all mappings among them preserving the infinite joins and the finite meets, which is called frame homomorphisms[2].

[1] If the least upper bound of the directed subset P is given by the union of the elements of P, we designate it by $\bigcup^{\uparrow} P$.

[2] This definition is adopted in the theory of the Stone duality, which is different from a convention that $\mathbf{cHey}$ denotes the category the arrows of which consist of the frame homomorphisms preserving the operation of implication.

This category is rather related to the category $\mathbf{Top}$ of topological spaces and continuous functions. Indeed, we have a contravariant functor $\mathbf{\Omega} : \mathbf{Top} \to \mathbf{cHey}$ which assigns to an object $\langle X, \mathcal{O}X \rangle$ of $\mathbf{Top}$ the complete Heyting algebra $\langle \mathcal{O}X, \subseteq \rangle$ of its open sets equipped with the order of set inclusion, and to an arrow $f : X \to Y$ in $\mathbf{Top}$ the inverse image function $\mathbf{\Omega} f : \mathbf{\Omega} Y \to \mathbf{\Omega} X$, namely $(\mathbf{\Omega} f)(U) = f^{-1}(U)$ for every $U \in \mathbf{\Omega} Y$. To see the reverse direction, suppose $\langle L, \sqsubseteq \rangle$ is an object of $\mathbf{cHey}$. Then we say that a filter F on L is completely prime if we have $F \cap P \neq \emptyset$ for every subset P of L such that $\bigsqcup P \in F$, and define $\mathsf{pt}\, L$ to be the set of all completely prime filters on L. Furthermore, on the set $\mathsf{pt}\, L$, we generate the topology $\mathcal{O}_{\mathsf{pt}} L$ by letting every open set be of the form $\{F \in \mathsf{pt}\, L \mid a \in F\}$ for some $a \in L$. This enables us to have a contravariant functor $\mathsf{pt} : \mathbf{cHey} \to \mathbf{Top}$ which assigns to an object $\langle L, \sqsubseteq \rangle$ of $\mathbf{cHey}$ the topological space $\langle \mathsf{pt}\, L, \mathcal{O}_{\mathsf{pt}} L \rangle$, and to an arrow $f : K \to L$ in $\mathbf{cHey}$ the continuous function $\mathsf{pt}\, f : \mathsf{pt}\, L \to \mathsf{pt}\, K$ defined by $(\mathsf{pt}\, f)(F) = f^{-1}(F)$ for every $F \in \mathsf{pt}\, L$.

Restricting appropriately the category $\mathbf{Top}$ to a full subcategory $\mathbf{C}$ of sober spaces and the category $\mathbf{cHey}$ to a full subcategory $\mathbf{D}$ of spatial lattices, the functors above establish a dual equivalence between $\mathbf{C}$ and $\mathbf{D}$. This categorical equivalence is so-called the Stone duality, in which for every object X of $\mathbf{C}$, we have a homeomorphism $\eta_X : X \to \mathsf{pt}(\mathbf{\Omega} X)$ in $\mathbf{C}$ by $\eta_X(a) = \{U \in \mathbf{\Omega} X \mid a \in U\}$ for every $a \in X$, and for every object L of $\mathbf{D}$, an order isomorphism $\varepsilon_L : L \to \mathbf{\Omega}(\mathsf{pt} L)$ in $\mathbf{D}$ by $\varepsilon_L(a) = \{F \in \mathsf{pt} L \mid a \in F\}$ for every $a \in L$. As an instance of the Stone duality, the discussion in this paper specifically make use of the dual equivalence between the category $\mathbf{Alg}$ as a full subcategory of $\mathbf{Top}$ and the category $\mathbf{CDA}$ as a full subcategory of $\mathbf{cHey}$.

3. A topological generalization of Sobolev's framework

Throughout this paper, we assume that the syntax of the formal system $\mathbf{NJ_2}$ is fixed as follows. The set $\mathbf{Prop_2}$ of the propositions of $\mathbf{NJ_2}$ is generated by the following abstract grammar:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \forall p.A \mid \exists p.A$$

where p ranges over the set $\mathbf{Vars}$ of propositional variables. We use letters $p, q, r, \ldots$ to denote propositional variables and $A, B, C, \ldots$ to denote propositions. For every $\Gamma \subseteq \mathbf{Prop_2}$ and $A \in \mathbf{Prop_2}$, we write $\Gamma \vdash_{\mathbf{NJ_2}} A$ if there exists a finite subset Γ_0 of Γ such that the judgement $\Gamma_0 \vdash A$ is derived by means of the inference rules of $\mathbf{NJ_2}$, see [3, Definition 11.1.2].

As a known framework of semantics for $\mathbf{NJ_2}$, we can find the Kripke models introduced by Sobolev [4], for which $\mathbf{NJ_2}$ is shown to be sound and complete. We begin with a brief review of its presentation. Let $\langle C, \leq \rangle$ be a partially ordered set of possible worlds, in which with every world $a \in C$ we associate its domain $d(a)$ as a set of upward-closed subsets of C endowed with the nested structure such that

$$a \leq b \implies d(a) \subseteq d(b)$$

for every $a, b \in C$. Then, the triple $\mathscr{C} = \langle C, \leq, d \rangle$ is said to be a Kripke model. We also say that a mapping ξ is an environment on $\mathscr{C}$ if $\xi(p)$ is an upward-closed subset of C for every $p \in \mathbf{Vars}$. For every Kripke model $\mathscr{C} = \langle C, \leq, d \rangle$, $a \in C$ and environment ξ, the forcing relation $a, \xi \Vdash_{\mathscr{C}} A$ is defined according to the structure of the proposition A, as follows:

1. $a, \xi \Vdash_{\mathscr{C}} \bot$ never holds.
2. $a, \xi \Vdash_{\mathscr{C}} p \iff a \in \xi(p)$.
3. $a, \xi \Vdash_{\mathscr{C}} A \wedge B \iff a, \xi \Vdash_{\mathscr{C}} A$ and $a, \xi \Vdash_{\mathscr{C}} B$.
4. $a, \xi \Vdash_{\mathscr{C}} A \vee B \iff a, \xi \Vdash_{\mathscr{C}} A$ or $a, \xi \Vdash_{\mathscr{C}} B$.
5. $a, \xi \Vdash_{\mathscr{C}} A \to B \iff \forall b \geq a\ (b, \xi \Vdash_{\mathscr{C}} A$ implies $b, \xi \Vdash_{\mathscr{C}} B)$.
6. $a, \xi \Vdash_{\mathscr{C}} \forall p.A \iff \forall b \geq a\ \forall U \in d(b) \quad b, \xi(p\!:\!U) \Vdash_{\mathscr{C}} A$.
7. $a, \xi \Vdash_{\mathscr{C}} \exists p.A \iff \exists U \in d(a) \quad a, \xi(p\!:\!U) \Vdash_{\mathscr{C}} A$.

We also write $a, \xi \Vdash_{\mathscr{C}} \Gamma$ for $\Gamma \subseteq \mathbf{Prop_2}$ if $a, \xi \Vdash_{\mathscr{C}} A$ holds for every $A \in \Gamma$. Determining the validity of a judgement in a Kripke model $\mathscr{C}$, we restrict ourselves only to the worlds and the environments which are admissible with respect to the propositions appearing in the judgement. Therefore, we write $\Gamma \models_{\mathscr{C}} A$ if

$$a, \xi \Vdash_{\mathscr{C}} \Gamma \implies a, \xi \Vdash_{\mathscr{C}} A$$

holds for every a and ξ such that $\xi(\mathrm{FV}(\Gamma, A)) \subseteq d(a)$. We further focus our attention on a specific type of Kripke model in which the condition

$$\exists U \in d(a)\ \ \forall b \geq a\ \ (b \in U \iff b, \xi \Vdash_{\mathscr{C}} A)$$

is satisfied for every a, ξ and A such that $\xi(\mathrm{FV}(A)) \subseteq d(a)$. Such Kripke models are said to be full, for which we obtain the statement of soundness and completeness, that is, a statement $\Gamma \vdash_{\mathbf{NJ_2}} A$ holds if and only if $\Gamma \models_{\mathscr{C}} A$ for every full Kripke model $\mathscr{C}$.

We note that this result by Sobolev deeply depends on the set of upward-closed subsets on a partially ordered set, which is comparable with a specific type of topology, the so-called Alexandrov topology. Incidentally, the Alexandrov topology on a poset L is defined to be the set of all the upward-closed subsets of L. By this analogy, we can regard the semantics by Sobolev as a version of neighbourhood semantics. Indeed, adopting a general topology instead of the Alexandrov topology induces a more general version of semantics, which we develop in the rest of this section. In particular, we will demonstrate that the soundness property is still ensured through this kind of generalization.

We suppose $\mathbf{C}$ is a subcategory of $\mathbf{Top}$ and $\langle X, \mathcal{O}X\rangle$ is an object of $\mathbf{C}$. With every element $a \in X$ of the space, we associate its domain $d(a)$ as a subset of the topology $\mathcal{O}X$ endowed with the nested structure such that

$$\exists U \in F_a\ \ \forall b \in U\ \ d(a) \subseteq d(b)$$

for every $a \in X$. Here, we denote the set of open neighbourhoods of a by F_a. We specify a neighbourhood of a that satisfies the condition above and denote it by U_a. Then we obtain the following lemma, which says roughly the openness of the set of admissible worlds.

Lemma 3.1. For every $K \subseteq \mathcal{O}X$, we have $\{a \in X \mid K \subseteq d(a)\} \in \mathcal{O}X$.

Proof:
Suppose $K \subseteq d(a)$. Then $K \subseteq d(b)$, for every $b \in U_a$. Hence, we obtain $\{a \in X \mid K \subseteq d(a)\} = \bigcup\{U_a \in \mathcal{O}X \mid K \subseteq d(a)\} \in \mathcal{O}X$. □

We say that the triple $\mathscr{X} = \langle X, \mathcal{O}X, d\rangle$ is a $\mathbf{C}$-neighbourhood model. We also say that a mapping ξ is an environment on $\mathscr{X}$ if $\xi(p) \in \mathcal{O}X$ for every $p \in \mathbf{Vars}$. For every $\mathbf{C}$-neighbourhood model $\mathscr{X} = \langle X, \mathcal{O}X, d\rangle, a \in X$ and environment ξ, we define the forcing relation $a, \xi \Vdash_{\mathscr{X}} A$ by induction on the structure of the proposition A, as follows:

1. $a, \xi \Vdash_{\mathscr{X}} \bot$ never holds.
2. $a, \xi \Vdash_{\mathscr{X}} p \iff a \in \xi(p)$.
3. $a, \xi \Vdash_{\mathscr{X}} A \wedge B \iff a, \xi \Vdash_{\mathscr{X}} A$ and $a, \xi \Vdash_{\mathscr{X}} B$.
4. $a, \xi \Vdash_{\mathscr{X}} A \vee B \iff a, \xi \Vdash_{\mathscr{X}} A$ or $a, \xi \Vdash_{\mathscr{X}} B$.
5. $a, \xi \Vdash_{\mathscr{X}} A \to B \iff \exists U \in F_a\ \forall b \in U\ (b, \xi \Vdash_{\mathscr{X}} A$ implies $b, \xi \Vdash_{\mathscr{X}} B)$.
6. $a, \xi \Vdash_{\mathscr{X}} \forall p.A \iff \exists U \in F_a\ \forall b \in U\ \forall V \in d(b)\quad b, \xi(p{:}V) \Vdash_{\mathscr{X}} A$.
7. $a, \xi \Vdash_{\mathscr{X}} \exists p.A \iff \exists V \in d(a)\quad a, \xi(p{:}V) \Vdash_{\mathscr{X}} A$.

Similarly to the definition by Sobolev, we write $a, \xi \Vdash_{\mathscr{X}} \Gamma$ for every $\Gamma \subseteq \mathbf{Prop_2}$ if $a, \xi \Vdash_{\mathscr{X}} A$ holds for every $A \in \Gamma$. We also write $\Gamma \models_{\mathscr{X}} A$ if

$$a, \xi \Vdash_{\mathscr{X}} \Gamma \implies a, \xi \Vdash_{\mathscr{X}} A$$

holds for every a and ξ such that $\xi(\mathrm{FV}(\Gamma, A)) \subseteq d(a)$. We say that a $\mathbf{C}$-neighbourhood model $\mathscr{X} = \langle X, \mathcal{O}X, d\rangle$ is full if the condition

$$\exists U \in d(a)\ \exists V \in F_a\ \forall b \in V\ \ (b \in U \Longleftrightarrow b, \xi \Vdash_{\mathscr{X}} A)$$

is satisfied for every proposition A, $a \in X$ and ξ such that $\xi(\mathrm{FV}(A)) \subseteq d(a)$. This condition forces every domain to include representatives of the proposition A, that is the open set U in the statement above. We specify one of such representatives found in $d(a)$ and denote it by $[\![A]\!]^a_\xi$. It is also clear from the condition of fullness that every domain is non-empty for every full neighbourhood model. Then, we write $\Gamma \models_{\mathbf{C}} A$ if we have $\Gamma \models_{\mathscr{X}} A$ for every full $\mathbf{C}$-neighbourhood model $\mathscr{X}$.

We write $\mathbf{Alex}$[3] for the category of the Alexandrov spaces, that is, the class of its objects consists of all posets endowed with the Alexandrov topology. From the viewpoint of our neighbourhood semantics, the Kripke models reported by Sobolev can be comprehended as $\mathbf{Alex}$-neighbourhood models. Thus, it is clear that the completeness of $\mathbf{NJ_2}$ is still ensured with respect to the validity based on the full $\mathbf{Top}$-neighbourhood models. On the other hand, we can verify the soundness as follows.

In the proof below, we assume $[\![A]\!]_\xi = \{a \in X \mid a, \xi \Vdash_{\mathscr{X}} A\}$ for every $A \in \mathbf{Prop_2}$ and $[\![\Gamma]\!]_\xi = \bigcap_{A\in\Gamma} [\![A]\!]_\xi$ for every $\Gamma \subseteq \mathbf{Prop_2}$. Then the following lemma says the openness of the set of worlds satisfying a proposition.

[3]Due to maintaining the categorical equivalence, it is appropriate to adopt the approximable relations [13, Definition 2.2.27] among posets as the arrows of $\mathbf{Alex}$. We omit its definition here because our discussion does not depend on the arrow part of the category.

Lemma 3.2. For every $\mathbf{Top}$-neighbourhood model $\mathscr{X} = \langle X, \mathcal{O}X, d\,\rangle$, $A \in \mathbf{Prop_2}$ and environment ξ, we have the following:
(1) If $a, \xi \Vdash_{\mathscr{X}} A$, then there exists $U \in F_a$ such that $b, \xi \Vdash_{\mathscr{X}} A$ for every $b \in U$.
(2) $[\![A]\!]_\xi \in \mathcal{O}X$.

Proof:
(1) By induction on the structure of A, we have the following cases:

Case 1: Suppose $A \equiv \bot$. Then, the statement is trivial.

Case 2: Suppose $A \equiv p$ and $a, \xi \Vdash_{\mathscr{X}} p$. Then, we have $\xi(p) \in F_a$, for which $b, \xi \Vdash_{\mathscr{X}} p$ is clear for every $b \in \xi(p)$.

Case 3: Suppose $A \equiv B \wedge C$ and $a, \xi \Vdash_{\mathscr{X}} B \wedge C$. Then, we have both $a, \xi \Vdash_{\mathscr{X}} B$ and $a, \xi \Vdash_{\mathscr{X}} C$. By the induction hypothesis, we can find $U, V \in F_a$ such that $b, \xi \Vdash_{\mathscr{X}} B$ and $c, \xi \Vdash_{\mathscr{X}} C$ for every $b \in U$ and $c \in V$, respectively. Hence, taking $U \cap V \in F_a$ completes the proof of this case.

Case 4: Suppose $A \equiv B \vee C$ and $a, \xi \Vdash_{\mathscr{X}} B \vee C$. Then, we have either $a, \xi \Vdash_{\mathscr{X}} B$ or $a, \xi \Vdash_{\mathscr{X}} C$. In the former case, the induction hypothesis allows us to have $U \in F_a$ such that $b, \xi \Vdash_{\mathscr{X}} B$ for every $b \in U$. Thus, $b, \xi \Vdash_{\mathscr{X}} B \vee C$ also follows for every $b \in U$. The latter case is proved in the same way.

Case 5: Suppose $A \equiv B \to C$ and $a, \xi \Vdash_{\mathscr{X}} B \to C$. Then, we have $U \in F_a$ such that $b, \xi \Vdash_{\mathscr{X}} B$ implies $b, \xi \Vdash_{\mathscr{X}} C$ for every $b \in U$. It is clear for this open neighbourhood U that $b, \xi \Vdash_{\mathscr{X}} B \to C$ for every $b \in U$.

Case 6: Suppose $A \equiv \forall p.B$ and $a, \xi \Vdash_{\mathscr{X}} \forall p.B$. Then, we have $U \in F_a$ such that $b, \xi(p : V) \Vdash_{\mathscr{X}} B$ for every $b \in U$ and $V \in d(b)$. It is clear for this open neighbourhood U that $b, \xi \Vdash_{\mathscr{X}} \forall p.B$ for every $b \in U$.

Case 7: Suppose $A \equiv \exists p.B$ and $a, \xi \Vdash_{\mathscr{X}} \exists p.B$. Then, we have $a, \xi(p : U) \Vdash_{\mathscr{X}} B$ for some $U \in d(a)$. By the induction hypothesis, we can find $V \in F_a$ such that $b, \xi(p : U) \Vdash_{\mathscr{X}} B$ for every $b \in V$. Taking $V \cap U_a \in F_a$, we obtain $U \in d(a) \subseteq d(b)$ and $b, \xi(p : U) \Vdash_{\mathscr{X}} B$ for every $b \in V \cap U_a$. Therefore, $b, \xi \Vdash_{\mathscr{X}} \exists p.B$ follows for every $b \in V \cap U_a$.

(2) It is clear from (1) that $[\![A]\!]_\xi = \bigcup\{U \in \mathcal{O}X \mid U \subseteq [\![A]\!]_\xi\} \in \mathcal{O}X$. □

From the preceding lemma, we know that the definition of the forcing relation concerning the existential quantifier can be presented in the same manner as the definition concerning the universal quantifier. This fact enables us to define an interpretation of the existential quantifier in our lattice models, which is discussed in Section 5.

Corollary 3.3. For every $\mathbf{Top}$-neighbourhood model $\mathscr{X} = \langle X, \mathcal{O}X, d\rangle$, we have $a, \xi \Vdash_{\mathscr{X}} \exists p.A$ if and only if

$$\exists U \in F_a \;\; \forall b \in U \;\; \exists V \in d(b) \;\; b, \xi(p : V) \Vdash_{\mathscr{X}} A.$$

Proof:
The "if" part is trivial. To see the "only-if" part, assume that $a, \xi(p : U) \Vdash_{\mathscr{X}} A$ for some $U \in d(a)$. Then, by Lemma 3.2 (1) we can find an open neighbourhood $V \in F_a$ such that $b, \xi(p : U) \Vdash_{\mathscr{X}} A$ for every $b \in V$. Therefore, taking $V \cap U_a \in F_a$, we obtain $U \in d(a) \subseteq d(b)$ and $b, \xi(p : U) \Vdash_{\mathscr{X}} A$ for every $b \in V \cap U_a$. □

To prove the soundness, for every domain we need a kind of approximation of the interpretation of substitution instances for the propositional variables, the existence of which is guaranteed by the fullness condition. This is summarized in the following lemma.

Lemma 3.4. Let $\mathscr{X} = \langle X, \mathcal{O}X, d\rangle$ be a full $\mathbf{Top}$-neighbourhood model. For every $A, B \in \mathbf{Prop_2}$, $a \in X$ and ξ such that $\xi(\mathrm{FV}(A[p := B])) \subseteq d(a)$, we have $a, \xi(p : [\![B]\!]^a_\xi) \Vdash_{\mathscr{X}} A$ if and only if $a, \xi \Vdash_{\mathscr{X}} A[p := B]$.

Proof:
A proof is by induction on the structure of A. In particular, for the case of $A \equiv p$, we can verify the statement by means of the equivalence between $a \in [\![B]\!]^a_\xi$ and $a, \xi \Vdash_{\mathscr{X}} B$. □

Theorem 3.5. If $\Gamma \vdash_{\mathbf{NJ_2}} A$, then $\Gamma \models_{\mathbf{Top}} A$.

Proof:
From the assumption, we have a judgement $\Gamma_0 \vdash A$ derivable in $\mathbf{NJ_2}$ for some finite subset Γ_0 of Γ. Therefore, it suffices to show that $\Gamma_0 \models_{\mathscr{X}} A$ holds for every full $\mathbf{Top}$-neighbourhood model $\mathscr{X} = \langle X, \mathcal{O}X, d\rangle$. We prove this by induction on the length of the derivation, where we omit some simpler cases. In this proof, we should pay attention to Cases 4 and 5, to which the existence of a representative of every proposition ensured by the fullness is indispensable.

Case 1: Suppose the last step of the derivation is given by Rule ($\to$I), that yields $\Gamma_0 \vdash A \to B$ from $\Gamma_0, A \vdash B$. We also assume that $a \in X$ and ξ satisfy $\xi(\mathrm{FV}(\Gamma_0, A \to B)) \subseteq d(a)$ and $a, \xi \Vdash_{\mathscr{X}} \Gamma_0$. Then, we have an open neighbourhood $[\![\Gamma_0]\!]_\xi \cap \{x \mid \xi(\mathrm{FV}(\Gamma_0, A \to B)) \subseteq d(x)\} \in F_a$ by Lemmas 3.1 and 3.2. Let us take a possible world $b \in [\![\Gamma_0]\!]_\xi \cap \{x \mid \xi(\mathrm{FV}(\Gamma_0, A \to B)) \subseteq d(x)\}$ such that $b, \xi \Vdash_{\mathscr{X}} A$. Then we obtain $b, \xi \Vdash_{\mathscr{X}} \Gamma_0, A$, which together with the inclusion $\xi(\mathrm{FV}(\Gamma_0, A, B)) \subseteq d(b)$ implies $b, \xi \Vdash_{\mathscr{X}} B$ by the induction hypothesis. This concludes that $a, \xi \Vdash_{\mathscr{X}} A \to B$.

Case 2: Suppose the last step of the derivation is given by Rule ($\to$E), that yields $\Gamma_0 \vdash B$ from $\Gamma_0 \vdash A \to B$ and $\Gamma_0 \vdash A$. We also assume that $a \in X$ and ξ satisfy $\xi(\mathrm{FV}(\Gamma_0, B)) \subseteq d(a)$ and $a, \xi \Vdash_{\mathscr{X}} \Gamma_0$. Then, we can take an environment ξ' such that $\xi'(p) = \xi(p)$ for every $p \in \mathrm{FV}(\Gamma_0, B)$ and $\xi'(p) \in d(a)$ for every $p \in \mathrm{FV}(A) \setminus \mathrm{FV}(\Gamma_0, B)$, for which $\xi'(\Gamma_0, A \to B) \subseteq d(a)$ and $a, \xi' \Vdash_{\mathscr{X}} \Gamma_0$ are obtained straightforwardly. Thus, we have $a, \xi' \Vdash_{\mathscr{X}} A \to B$ and $a, \xi' \Vdash_{\mathscr{X}} A$ by the induction hypothesis, from which $a, \xi' \Vdash_{\mathscr{X}} B$ follows immediately. This concludes that $a, \xi \Vdash_{\mathscr{X}} B$.

Case 3: Suppose the last step of the derivation is given by Rule ($\forall$I), that yields $\Gamma_0 \vdash \forall p.A$ from $\Gamma_0 \vdash A$ where $p \notin \mathrm{FV}(\Gamma_0)$. We also assume that $a \in X$ and ξ satisfy $\xi(\mathrm{FV}(\Gamma_0, \forall p.A)) \subseteq d(a)$ and $a, \xi \Vdash_{\mathscr{X}} \Gamma_0$. Then we take an open neighborhood $[\![\Gamma_0]\!]_\xi \cap \{x \mid \xi(\mathrm{FV}(\Gamma_0, \forall p.A)) \subseteq d(x)\} \in F_a$. Let us take a possible world $b \in [\![\Gamma_0]\!]_\xi \cap \{x \mid \xi(\mathrm{FV}(\Gamma_0, \forall p.A)) \subseteq d(x)\}$ and $V \in d(b)$. Then, we obtain $b, \xi(p : V) \Vdash_{\mathscr{X}} \Gamma_0$ since Γ_0 contains no free occurrence of the variable p. This together with the inclusion of $\xi(p : V)(\mathrm{FV}(\Gamma_0, A)) \subseteq d(b)$ implies $b, \xi(p : V) \Vdash_{\mathscr{X}} A$ by the induction hypothesis. Therefore, we conclude that $a, \xi \Vdash_{\mathscr{X}} \forall p.A$.

Case 4: Suppose the last step of the derivation is given by Rule ($\forall$E), that yields $\Gamma_0 \vdash A[p := B]$ from $\Gamma_0 \vdash \forall p.A$. We also assume that $a \in X$ and ξ satisfy $\xi(\mathrm{FV}(\Gamma_0, A[p := B])) \subseteq d(a)$ and $a, \xi \Vdash_{\mathscr{X}} \Gamma_0$. Then, we have $\xi(\mathrm{FV}(\Gamma_0, \forall p.A)) \subseteq d(a)$ and obtain $a, \xi \Vdash_{\mathscr{X}} \forall p.A$ by the induction hypothesis. On the other hand, we have $[\![B]\!]^a_\xi \in d(a)$ because of the fullness of the model $\mathscr{X}$, from which $a, \xi(p : [\![B]\!]^a_\xi) \Vdash_{\mathscr{X}} A$ follows. Hence, we conclude that $a, \xi \Vdash_{\mathscr{X}} A[p := B]$ by Lemma 3.4.

Case 5: Suppose the last step of the derivation is given by Rule (∃I), that yields $\Gamma_0 \vdash \exists p.A$ from $\Gamma_0 \vdash A[p := B]$. We also assume that $a \in X$ and ξ satisfy $\xi(\mathrm{FV}(\Gamma_0, \exists p.A)) \subseteq d(a)$ and $a, \xi \Vdash_{\mathscr{X}} \Gamma_0$. Then, we can take an environment ξ' such that $\xi'(p) = \xi(p)$ for every $p \in \mathrm{FV}(\Gamma_0, \exists p.A)$ and $\xi'(p) \in d(a)$ for every $p \in \mathrm{FV}(B) \setminus \mathrm{FV}(\Gamma_0, \exists p.A)$, for which $\xi'(\Gamma_0, A[p := B]) \subseteq d(a)$ and $a, \xi' \Vdash_{\mathscr{X}} \Gamma_0$ are verified easily. Thus, $a, \xi' \Vdash_{\mathscr{X}} A[p := B]$ follows from the induction hypothesis. On the other hand, we have $[\![B]\!]^a_\xi \in d(a)$ because of the fullness of the model $\mathscr{X}$, for which $a, \xi'(p : [\![B]\!]^a_\xi) \Vdash_{\mathscr{X}} A$ holds by Lemma 3.4. This enables us to have $a, \xi' \Vdash_{\mathscr{X}} \exists p.A$, so that $a, \xi \Vdash_{\mathscr{X}} \exists p.A$.

Case 6: Suppose the last step of the derivation is given by Rule (∃E), that yields $\Gamma_0 \vdash B$ from $\Gamma_0 \vdash \exists p.A$ and $\Gamma_0, A \vdash B$ where $p \notin \mathrm{FV}(\Gamma_0, B)$. We also assume that $a \in X$ and ξ satisfy $\xi(\mathrm{FV}(\Gamma_0, B)) \subseteq d(a)$ and $a, \xi \Vdash_{\mathscr{X}} \Gamma_0$. Then, we can take an environment ξ' such that $\xi'(p) = \xi(p)$ for every $p \in \mathrm{FV}(\Gamma_0, B)$ and $\xi'(p) \in d(a)$ for every $p \in \mathrm{FV}(\exists p.A) \setminus \mathrm{FV}(\Gamma_0, B)$, for which $\xi'(\mathrm{FV}(\Gamma_0, \exists p.A)) \subseteq d(a)$ and $a, \xi' \Vdash_{\mathscr{X}} \Gamma_0$ are verified easily. Thus, $a, \xi' \Vdash_{\mathscr{X}} \exists p.A$ follows from the induction hypothesis, so we have $a, \xi'(p : U) \Vdash_{\mathscr{X}} A$ for some $U \in d(a)$. This allows us to have $\xi'(p : U)(\mathrm{FV}(\Gamma_0, A, B)) \subseteq d(a)$ and $a, \xi'(p : U) \Vdash_{\mathscr{X}} \Gamma_0, A$ since Γ_0 contains no free occurrence of the variable p. Therefore, $a, \xi'(p : U) \Vdash_{\mathscr{X}} B$ by the induction hypothesis, so that $a, \xi \Vdash_{\mathscr{X}} B$ since B contains no free occurrence of the variable p. □

The soundness of $\mathbf{NJ_2}$ is an essential property to be verified for our semantics, which underlies the discussion in Section 5. In contrast, the converse of Theorem 3.5 is immediate from Sobolev's completeness theorem [3, 4]. This is because Sobolev's canonical model $\mathscr{S} = \langle S, \leq, d \rangle$ introduced in his proof of the completeness can be regarded as an instance of **Top**-neighbourhood models. We also note that the model $\mathscr{S}$ gives a typical example of Kripke models in which domains vary depending on the corresponding world.

4. Stone duality for Alexandrov spaces

To give a lattice-theoretical counterpart of our neighbourhood models, we depend on a correspondence between topological spaces and complete Heyting algebras. Actually, it is well known as the Stone duality [14] that a dual equivalence is established generally among various categories of sober spaces and those of spatial lattices. In this regard, we are able to employee a modified form of the duality despite that Alexandrov spaces underlying our model construction are not always sober. More precisely, we establish a dual equivalence between the category **Alex** and the category **CDA** of the completely distributive algebraic lattices and the frame homomorphisms.

This is accomplished through an intermediate framework, the category **Alg** of the algebraic domains equipped with the Scott topology and the continuous functions. To see its definition, let $\langle L, \sqsubseteq \rangle$ be a complete poset. Then we define the Scott topology $\mathcal{O}_S L$ on L to be the set of all upward-closed subsets of L which are inaccessible by directed joins, that is, every Scott-open set $U \in \mathcal{O}_S L$ satisfies $U \cap P \neq \emptyset$ for every directed subset $P \subseteq L$ such that $\bigsqcup^{\uparrow} P \in U$. We say that an element $a \in L$ is compact if the principal filter $\uparrow a = \{ b \in L \mid a \sqsubseteq b \}$ is Scott open, and write $\mathsf{K}L$ for the set of all compact elements of L. Then the complete poset L is said to be an algebraic domain if for every $a \in L$, the set $\{x \in \mathsf{K}L \mid x \sqsubseteq a\}$ is directed and its least upper bound coincides with a.

Among these categories, we have four functors as the diagram below, and the rest of this section is mainly devoted to see their details, in which we restrict our attention only to the objects of the categories since our model construction does not depend on any information about arrows. So, for categories $\mathbf{C}$ and $\mathbf{D}$, we simply write $\mathbf{C} \subseteq \mathbf{D}$ to mean that the class of all objects of $\mathbf{D}$ includes that of $\mathbf{C}$, and $X \in \mathbf{C}$ to mean that X is an object of the category $\mathbf{C}$.

$$\mathbf{Alex} \underset{\mathsf{K}}{\overset{\mathsf{Idl}}{\rightleftarrows}} \mathbf{Alg} \underset{\mathsf{pt}}{\overset{\Omega}{\rightleftarrows}} \mathbf{CDA}$$

Suppose $X \in \mathbf{Alex}$. Then X is endowed with a partial order relation underlying the Alexandrov topology $\mathcal{O}_{\mathcal{A}}X$[4], by which we generate the set $\mathsf{Idl}\ X$ of all ideals over X. It is well known that the ideal completion of any poset is an algebraic domain with respect to the order of set inclusion. This together with the Scott topology $\mathcal{O}_{\mathcal{S}}(\mathsf{Idl}\ X)$ allows us to have $\mathsf{Idl}\ X \in \mathbf{Alg}$. So this mapping from $\mathbf{Alex}$ to $\mathbf{Alg}$ defines the object part of the functor Idl.

The translation in the opposite direction can be defined for a more general class of topological spaces. To see it, suppose $\langle X, \mathcal{O}X\rangle$ is a sober space. Then X is regarded as a complete poset by the specialization order[5] based on the topology $\mathcal{O}X$, as is explained in [13, Chapter 7]. So we naively restrict it to the poset $\mathsf{K}X$ of the compact elements of X, on which we generate the relative topology $(\mathcal{O}X)_{\mathsf{K}X} = \{U \cap \mathsf{K}X \mid U \in \mathcal{O}X\}$. This construction defines the object part of the functor K.

Restricting our attention to a topological space $\langle X, \mathcal{O}_{\mathcal{S}}X\rangle$ in $\mathbf{Alg}$ defined over an algebraic domain $\langle X, \sqsubseteq\rangle$, we know that the partial order $\sqsubseteq$ coincides with the specialization order based on the Scott topology $\mathcal{O}_{\mathcal{S}}X$. So, as the result of applying the functor K to the space $\langle X, \mathcal{O}_{\mathcal{S}}X\rangle$, we obtain the topological space $\langle \mathsf{K}X, (\mathcal{O}_{\mathcal{S}}X)_{\mathsf{K}X}\rangle$ based on the partial order $\sqsubseteq$. This relative topology $(\mathcal{O}_{\mathcal{S}}X)_{\mathsf{K}X}$ is shown to be identical with the Alexandrov topology $\mathcal{O}_{\mathcal{A}}(\mathsf{K}X)$ which is also generated based on the partial order $\sqsubseteq$, and so we have $\langle \mathsf{K}X, (\mathcal{O}_{\mathcal{S}}X)_{\mathsf{K}X}\rangle \in \mathbf{Alex}$. These properties are shown in the following lemma.

Lemma 4.1. Suppose $\langle X, \mathcal{O}_{\mathcal{S}}X\rangle \in \mathbf{Alg}$ is defined over an algebraic domain $\langle X, \sqsubseteq\rangle$. Then, we have the following:
(1) For every $a, b \in X$, we have $a \sqsubseteq b$ if and only if $a \in U$ implies $b \in U$ for every $U \in \mathcal{O}_{\mathcal{S}}X$.
(2) We have $(\mathcal{O}_{\mathcal{S}}X)_{\mathsf{K}X} = \mathcal{O}_{\mathcal{A}}(\mathsf{K}X)$.

Proof:
(1) To see the "if" part, suppose $x \in \mathsf{K}X$ and $x \sqsubseteq a$. Then we know that $a \in {\uparrow} x \in \mathcal{O}_{\mathcal{S}}X$, from which $b \in {\uparrow} x$ follows by the assumption. Hence we obtain $a = \bigsqcup^{\uparrow}\{x \in \mathsf{K}X \mid x \sqsubseteq a\} \sqsubseteq b$ since X is an algebraic domain. The "only-if" part is clear from the definition of Scott-open set.

[4]We notationally distinguish between the Alexandrov topology and the Scott topology, and use the letter $\mathcal{O}_{\mathcal{A}}$ for the former and the letter $\mathcal{O}_{\mathcal{S}}$ for the latter.

[5]Every topological space $\langle X, \mathcal{O}X\rangle$ satisfying T_0-separation axiom can be equipped with a partial order relation by setting $a \sqsubseteq_{\mathcal{O}X} b$ if $a \in U$ implies $b \in U$ for every $U \in \mathcal{O}X$, which is called the specialization order based on the topology $\mathcal{O}X$.

(2) Because of (1), it suffices to verify that $(\mathcal{O}_{\mathcal{S}}X)_{\mathsf{K}X}$ is the set of all upward-closed subsets of $\mathsf{K}X$ with respect to the partial order $\sqsubseteq$. It is clear from the definition of Scott-open sets that $(\mathcal{O}_{\mathcal{S}}X)_{\mathsf{K}X} \subseteq \mathcal{O}_{\mathcal{A}}(\mathsf{K}X)$. So we have to show the opposite inclusion. For every $U \in \mathcal{O}_{\mathcal{A}}(\mathsf{K}X)$, we define

$$U^* = \{\, a \in X \mid \downarrow a \cap \mathsf{K}X \cap U \neq \emptyset\}$$

and show that $U^* \in \mathcal{O}_{\mathcal{S}}X$. To see the upward closedness, we suppose $a \in U^*$ and $a \sqsubseteq b$. Then, $\downarrow a \cap \mathsf{K}X \cap U \neq \emptyset$ and $\downarrow a \subseteq \downarrow b$ hold, from which we obtain $\downarrow b \cap \mathsf{K}X \cap U \neq \emptyset$. This implies that $b \in U^*$. To verify the inaccessibility by directed joins, we assume that P is a directed subset of X and $\bigsqcup^{\uparrow} P \in U^*$, namely $\downarrow (\bigsqcup^{\uparrow} P) \cap \mathsf{K}X \cap U \neq \emptyset$. Then, we can find $x \in \downarrow (\bigsqcup^{\uparrow} P) \cap \mathsf{K}X \cap U$, for which there exists $a \in P$ such that $x \sqsubseteq a$ because of the compactness of x. For this $a \in P$, we obtain $x \in \downarrow a \cap \mathsf{K}X \cap U$, from which $a \in U^*$ is immediate. Accordingly, we obtain $U^* \in \mathcal{O}_{\mathcal{S}}X$. Furthermore, we obtain $U = U^* \cap \mathsf{K}X$ as follows:

$$\begin{aligned} x \in U^* \cap \mathsf{K}X &\iff \downarrow x \cap \mathsf{K}X \cap U \neq \emptyset \ \ \& \ \ x \in \mathsf{K}X \\ &\iff x \in U \end{aligned}$$

This leads us to conclude that $\mathcal{O}_{\mathcal{A}}(\mathsf{K}X) \subseteq (\mathcal{O}_{\mathcal{S}}X)_{\mathsf{K}X}$. □

Next we turn our attention to the correspondence between $\mathbf{Alg}$ and $\mathbf{CDA}$. The dual equivalence of these categories is just an instance of the Stone duality explained in Section 2. According to it, for every $X \in \mathbf{Alg}$, the complete Heyting algebra $\langle \mathbf{\Omega}X, \subseteq \rangle$ is given by the poset of its open sets equipped with the order of set inclusion, which is known to be an object of $\mathbf{CDA}$, that is, a complete lattice which is algebraic and satisfies the complete distributivity law. As for the reverse direction, for every $L \in \mathbf{CDA}$, the poset $\langle \mathsf{pt}\, L, \subseteq \rangle$ is an algebraic domain and the Scott topology $\mathcal{O}_{\mathcal{S}}(\mathsf{pt}\, L)$ defined over this algebraic domain coincides with $\mathcal{O}_{\mathsf{pt}}L$. This ensures $\langle \mathsf{pt}\, L, \mathcal{O}_{\mathsf{pt}}L\rangle \in \mathbf{Alg}$.

For the compositions of these functors, we abbreviate the composition $\mathbf{\Omega} \circ \mathsf{Idl}$ to $\mathbf{\Omega}^*$, which assigns to an Alexandrov space X the completely distributive algebraic lattice $\langle \mathcal{O}_{\mathcal{S}}(\mathsf{Idl}\, X), \subseteq \rangle$. We also abbreviate the composition $\mathsf{K} \circ \mathsf{pt}$ to pt^*, which assigns, more generally, to a complete Heyting algebra L the topological space $\langle \mathsf{K}(\mathsf{pt}\, L), (\mathcal{O}_{\mathsf{pt}}L)_{\mathsf{K}(\mathsf{pt}L)}\rangle$[6]. Especially, in case where $\langle L, \sqsubseteq \rangle \in \mathbf{CDA}$, the result of applying the functor pt^* is ensured to be an Alexandrov space. Denoting the topology $(\mathcal{O}_{\mathsf{pt}}L)_{\mathsf{K}(\mathsf{pt}L)}$ in the following, we abbreviate it to $\mathcal{O}_{\mathsf{pt}^*}L$, in which we are to interpret all propositions of $\mathbf{NJ_2}$ in Section 5. When $\langle L, \sqsubseteq\rangle \in \mathbf{CDA}$, this topology is identical with $(\mathcal{O}_{\mathcal{S}}(\mathsf{pt}\, L))_{\mathsf{K}(\mathsf{pt}L)}$ as well as $\mathcal{O}_{\mathcal{A}}(\mathsf{K}(\mathsf{pt}L))$, and the poset $\langle \mathcal{O}_{\mathsf{pt}^*} L, \subseteq \rangle$ is shown to be isomorphic to $\langle L, \sqsubseteq\rangle$ because of the representation theorem [15, 16] of completely distributive algebraic lattices.

Suppose $X \in \mathbf{Alex}$. Then, it is necessary that X and $\mathsf{pt}^*(\mathbf{\Omega}^* X)$ are homeomorphic. Actually, we describe a homeomorphism between X and $\mathsf{pt}^*(\mathbf{\Omega}^* X)$ in the following, which underlies the construction of our lattice models. To this end, we begin with a characterization of the completely prime elements of $\mathbf{\Omega}^* X$ by the principal filters on $\mathsf{Idl}X$ especially of the form $\uparrow(\downarrow a) = \{I \in \mathsf{Idl}X \mid \downarrow a \subseteq I\}$ for some $a \in X$.

[6]It is well known that the functor pt always assigns a sober space to a complete Heyting algebra, to which we can apply the functor K.

Lemma 4.2. (1) We have $\uparrow(\downarrow a) \in \mathbf{\Omega}^* X$ for every $a \in X$.
(2) Given a subset U of $\mathsf{Idl}\ X$, we have that U is a completely prime element[7] of $\mathbf{\Omega}^* X$ if and only if $U = \uparrow(\downarrow a)$ for some $a \in X$.

Proof:
(1) Since $\uparrow(\downarrow a)$ is clearly upward closed, it suffices to show the inaccessibility by directed joins. Suppose $\downarrow a \subseteq \bigcup^{\uparrow}_{\lambda \in \Lambda} I_\lambda$ where $\{I_\lambda \mid \lambda \in \Lambda\}$ is directed subset of $\mathsf{Idl}\ X$. Then $a \in \downarrow a \subseteq \bigcup^{\uparrow}_{\lambda \in \Lambda} I_\lambda$ follows. So we can find an index $\lambda \in \Lambda$ such that $a \in I_\lambda$, for which $\downarrow a \subseteq I_\lambda$ is clear.

(2) To see the "if" part, let us suppose $U \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$ where $U_\lambda \in \mathbf{\Omega}^* X$ for every $\lambda \in \Lambda$. Then $\downarrow a \in \uparrow(\downarrow a) \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$ follows. So we can find an index $\lambda \in \Lambda$ such that $\downarrow a \in U_\lambda$, for which $U = \uparrow(\downarrow a) \subseteq U_\lambda$ is clear.

We next show the "only-if" part. Suppose $I \in U$. Then we have $\bigcup^{\uparrow}\{\downarrow a \mid \downarrow a \subseteq I\} = I \in U$. This ensures the existence of a principal ideal $\downarrow a$ such that $\downarrow a \subseteq I$ and $\downarrow a \in U$ because U is a Scott-open set on $\mathsf{Idl}\ X$. For the ideal I, it is clear that $I \in \bigcup\{\uparrow(\downarrow a) \in \mathbf{\Omega}^* X \mid \downarrow a \in U\}$. Accordingly, we conclude $U \subseteq \bigcup\{\uparrow(\downarrow a) \in \mathbf{\Omega}^* X \mid \downarrow a \in U\}$. By our assumption, we can find $\downarrow a \in U$ such that $U \subseteq \uparrow(\downarrow a)$, from which $U = \uparrow(\downarrow a)$ is immediate. □

It is well known as a general property of Stone duality explained in Section 2 that every element of $\mathsf{pt}(\mathbf{\Omega}^* X)$ is characterized as a set of open neighborhoods $\{U \in \mathbf{\Omega}^* X \mid I \in U\}$ for some $I \in \mathsf{Idl}\ X$, which we may denote by F_I according to the notation in Section 3. In addition to this fact, we verify that every compact element of $\mathsf{pt}(\mathbf{\Omega}^* X)$ is of the form $F_{\downarrow a}$ for some $a \in X$, which is identical with the principal filter $\{U \in \mathbf{\Omega}^* X \mid \uparrow(\downarrow a) \subseteq U\}$ on $\mathbf{\Omega}^* X$.

Lemma 4.3. We have $F_{\downarrow a} \in \mathsf{pt}^*(\mathbf{\Omega}^* X)$ for every $a \in X$.

Proof:
It suffices to verify the compactness of $F_{\downarrow a}$. Suppose $F_{\downarrow a} \subseteq \bigcup^{\uparrow}_{\lambda \in \Lambda} F_{I_\lambda}$ where $\{F_{I_\lambda} \mid \lambda \in \Lambda\}$ is a directed subset of $\mathsf{pt}(\mathbf{\Omega}^* X)$. Then $\uparrow(\downarrow a) \in \bigcup^{\uparrow}_{\lambda \in \Lambda} F_{I_\lambda}$ holds by Lemma 4.2. Then we can find an index $\lambda \in \Lambda$ such that $\uparrow(\downarrow a) \in F_{I_\lambda}$, for which $F_{\downarrow a} \subseteq F_{I_\lambda}$ is clear. □

As a simple example concerning these constructions, let us consider the set ω of natural numbers ordered by $0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq \cdots$, which, together with the Alexandrov topology $\mathcal{O}_{\mathcal{A}}\omega$, gives an object of $\mathbf{Alex}$. Then it follows that $\mathsf{K}(\mathsf{Idl}\ \omega) = \{\downarrow n \mid n \in \omega\}$ and $\mathsf{Idl}\ \omega = \mathsf{K}(\mathsf{Idl}\ \omega) \cup \{\omega\}$, on which the Scott topology is generated as the set $\{\uparrow(\downarrow n) \mid n \in \omega\} \cup \{\emptyset\}$. This set is ordered by the set inclusion as $\emptyset \subseteq \cdots \subseteq \uparrow(\downarrow 1) \subseteq \uparrow(\downarrow 0)$, by which we certainly obtain $\langle \mathbf{\Omega}^*\omega, \subseteq \rangle \in \mathbf{CDA}$. Applying the functor pt to $\mathbf{\Omega}^*\omega$, we further obtain that $\mathsf{pt}(\mathbf{\Omega}^*\omega) = \{F_{\downarrow n} \mid n \in \omega\} \cup \{F_\omega\}$ where F_ω is identical with the set $\bigcup_{n \in \omega} F_{\downarrow n} = \{\uparrow(\downarrow n) \mid n \in \omega\}$. The set $\mathsf{pt}(\mathbf{\Omega}^*\omega)$ is also ordered by the set inclusion as $F_{\downarrow 0} \subseteq F_{\downarrow 1} \subseteq \cdots \subseteq F_\omega$. Finally, by restricting it to the set of compact elements, we obtain the same structure $\mathsf{pt}^*(\mathbf{\Omega}^*\omega) = \{F_{\downarrow n} \mid n \in \omega\}$ as ω.

This observation naturally leads us to define a function $\upsilon : X \to \mathsf{pt}^*(\mathbf{\Omega}^* X)$ by $\upsilon(a) = F_{\downarrow a}$ for every $a \in X$. Then, by the following lemma, the mapping υ is verified to be a homeomorphism.

[7]For $L \in \mathbf{cHey}$, an element $a \in L$ is said to be completely prime if $\uparrow a \in \mathsf{pt}\,L$.

Lemma 4.4. (1) For every $a, b \in X$, $a \sqsubseteq b$ if and only if $\upsilon(a) \subseteq \upsilon(b)$.
(2) The function υ is bijective.
(3) We have $\mathcal{O}_{\mathsf{pt}^*}(\mathbf{\Omega}^* X) = \{\upsilon(U) \mid U \in \mathcal{O}_{\mathcal{A}} X\}$.

Proof:
(1) Suppose $\upsilon(a) \subseteq \upsilon(b)$. Then $\uparrow(\downarrow a) \in \upsilon(a) \subseteq \upsilon(b)$ by Lemma 4.2. This implies $\downarrow b \in\uparrow(\downarrow a)$ so that $\downarrow a \subseteq\downarrow b$. Conversely, suppose $\downarrow a \subseteq\downarrow b$ and $U \in \upsilon(a)$. Then we have $\downarrow b \in U$ because of the upward closedness of U, from which $U \in \upsilon(b)$ follows.

(2) It is immediate from (1) that υ is injective. To show that υ is surjective, we suppose F_I is a compact element of $\mathsf{pt}(\mathbf{\Omega}^* X)$ for some $I \in \mathsf{Idl}\ X$. We also assume that $U \in F_I$. Then we obtain $\bigcup^{\uparrow}\{\downarrow a \mid a \in I\} = I \in U$. Since U is Scott open, we can find $a \in I$ such that $\downarrow a \in U$, and so $U \in \upsilon(a)$. This is why we have $F_I \subseteq \upsilon(a)$ for some $a \in I$. For this element a, we also obtain $\upsilon(a) \subseteq F_I$ since $\downarrow a \subseteq I$. This completes the proof of $\mathsf{pt}^*(\mathbf{\Omega}^* X) = \upsilon(X)$.

(3) It is necessary from the definition that $\mathcal{O}_{\mathsf{pt}^*}(\mathbf{\Omega}^* X) = \mathcal{O}_{\mathcal{A}}(\mathsf{pt}^*(\mathbf{\Omega}^* X))$. By (1) and (2), we then conclude that the structure of topology is preserved under the function υ and its inverse.

Here we confirm the equality directly, as follows. By means of (2) and the definitions of $\mathbf{\Omega}^*$ and pt^*, every element of $\mathcal{O}_{\mathsf{pt}^*}(\mathbf{\Omega}^* X)$ is of the form $\{\upsilon(a) \in \mathsf{pt}^*(\mathbf{\Omega}^* X) \mid a \in X \,\&\, U \in \upsilon(a)\}$ for some $U \in \mathbf{\Omega}^* X$, and it is clearly an upward-closed subset of $\mathsf{pt}^*(\mathbf{\Omega}^* X)$. On the other hand, suppose $U \in \mathcal{O}_{\mathcal{A}}(\mathsf{pt}^*(\mathbf{\Omega}^* X))$. Then, defining $U^* \in \mathbf{\Omega}^* X$ to be $\{I \in \mathsf{Idl}\ X \mid \exists a \in X\ (\upsilon(a) \in U \,\&\, a \in I)\}$, we obtain

$$\upsilon(a) \in U \iff \downarrow a \in U^* \iff U^* \in \upsilon(a).$$

Therefore $U = \{\upsilon(a) \in \mathsf{pt}^*(\mathbf{\Omega}^* X) \mid a \in X \,\&\, U^* \in \upsilon(a)\} \in \mathcal{O}_{\mathsf{pt}^*}(\mathbf{\Omega}^* X)$ holds. □

5. Lattice models of the system $\mathbf{NJ}_2$

Based on the dual equivalence between $\mathbf{Alex}$ and $\mathbf{CDA}$, we introduce a lattice-theoretical counterpart of the $\mathbf{Alex}$-neighbourhood models. More generally, we can introduce a notion of lattice models naively by describing the definition of $\mathbf{Top}$-neighbourhood models in terms of complete Heyting algebras according to the functor pt^* studied in Section 4.

Let $\mathbf{C}$ be a category such that $\mathbf{C} \subseteq \mathbf{cHey}$ and $\langle L, \sqsubseteq \rangle \in \mathbf{C}$. Then, we define a framework of interpretation of $\mathbf{NJ}_2$ by means of the topological space $\langle \mathsf{pt}^* L, \mathcal{O}_{\mathsf{pt}^*} L \rangle$. Actually, we assume that d is a mapping such that $d(F) \subseteq \mathcal{O}_{\mathsf{pt}^*} L$ for every $F \in \mathsf{pt}^* L$. Then we say that the structure $\mathscr{L} = \langle L, \sqsubseteq, d \rangle$ is a $\mathbf{C}$-lattice model of $\mathbf{NJ}_2$ if, for every $F \in \mathsf{pt}^* L$, there exists an open neighbourhood $U \in \mathcal{O}_{\mathsf{pt}^*} L$ of F such that

$$\forall G \in U \;\; d(F) \subseteq d(G).$$

We define an environment ξ on $\mathscr{L}$ as a mapping from $\mathbf{Vars}$ to $\mathcal{O}_{\mathsf{pt}^*} L$. Then, for every proposition A, environment ξ and $\mathbf{C}$-lattice model $\mathscr{L}$, we define the interpretation $[\![A]\!]_\xi \in \mathcal{O}_{\mathsf{pt}^*} L$ of A under ξ by induction on the structure of A, as follows:

1. $[\![\bot]\!]_\xi = \emptyset$.
2. $[\![p]\!]_\xi = \xi(p)$.
3. $[\![A \wedge B]\!]_\xi = [\![A]\!]_\xi \cap [\![B]\!]_\xi$.
4. $[\![A \vee B]\!]_\xi = [\![A]\!]_\xi \cup [\![B]\!]_\xi$.
5. $[\![A \to B]\!]_\xi = \bigcup \{U \in \mathcal{O}_{\mathsf{pt}^*} L \mid [\![A]\!]_\xi \cap U \subseteq [\![B]\!]_\xi\}$.
6. $[\![\forall p.A]\!]_\xi = \bigcup \{U \in \mathcal{O}_{\mathsf{pt}^*} L \mid \forall F \in U \;\; \forall V \in d(F) \;\; F \in [\![A]\!]_{\xi(p:V)}\}$.
7. $[\![\exists p.A]\!]_\xi = \bigcup \{U \in \mathcal{O}_{\mathsf{pt}^*} L \mid \forall F \in U \;\; \exists V \in d(F) \;\; F \in [\![A]\!]_{\xi(p:V)}\}$.

We also define $[\![\Gamma]\!]_\xi = \bigcap_{A \in \Gamma} [\![A]\!]_\xi$ for every $\Gamma \subseteq \mathbf{Prop_2}$ and ξ.

By analogy with the definition concerning neighbourhood models, we introduce a notion of validity based on the lattice models. For a $\mathbf{C}$-lattice model $\mathscr{L} = \langle L, \sqsubseteq, d \rangle$, we write $\Gamma \models_{\mathscr{L}} A$ if

$$F \in [\![\Gamma]\!]_\xi \Longrightarrow F \in [\![A]\!]_\xi$$

holds for every $F \in \mathsf{pt}^* L$ and ξ such that $\xi(\mathrm{FV}(\Gamma, A)) \subseteq d(F)$. We also say that a $\mathbf{C}$-lattice model $\mathscr{L}$ is full if, for every A, ξ and $F \in \mathsf{pt}^* L$ such that $\xi(\mathrm{FV}(A)) \subseteq d(F)$, we can find $U \in d(F)$ and $V \in \mathcal{O}_{\mathsf{pt}^*} L$ which satisfy $F \in V$ and $U \cap V = [\![A]\!]_\xi \cap V$. Then, we write $\Gamma \models_{\mathbf{C}} A$ if we have $\Gamma \models_{\mathscr{L}} A$ for every full $\mathbf{C}$-lattice model $\mathscr{L}$.

This definition of the lattice models actually works in order to give a both-way translation between $\mathbf{Alex}$-neighbourhood models and $\mathbf{CDA}$-lattice models. Indeed, for every $\mathbf{Alex}$-neighbourhood model $\mathscr{X} = \langle X, \mathcal{O}_{\mathcal{A}} X, d \rangle$, we introduce an $\mathbf{CDA}$-lattice model $\mathbf{\Omega}^* \mathscr{X}$ by

$$\mathbf{\Omega}^* \mathscr{X} = \langle \mathbf{\Omega}^* X, \subseteq, \upsilon \circ d \circ \upsilon^{-1} \rangle,$$

and we may compare it with the model $\mathscr{X}$ in the sense that the forcing relation and the fullness on $\mathscr{X}$ is preserved in the model $\mathbf{\Omega}^* \mathscr{X}$. This fact leads us to obtain the completeness of $\mathbf{NJ}_2$ with respect to $\mathbf{CDA}$-lattice models.

Lemma 5.1. Suppose $\mathscr{X} = \langle X, \mathcal{O}_{\mathcal{A}} X, d \rangle$ is an $\mathbf{Alex}$-neighbourhood model. Then we have the following:
(1) For every A, ξ and $a \in X$, we have $a, \xi \Vdash_{\mathscr{X}} A$ if and only if $\upsilon(a) \in [\![A]\!]_{\upsilon \circ \xi}$ in $\mathbf{\Omega}^* \mathscr{X}$.
(2) If $\mathscr{X}$ is full, then $\mathbf{\Omega}^* \mathscr{X}$ is also full.

Proof:
(1) It is given by induction on the structure of A. Omitting proofs of trivial cases, we verify some cases below.

Case 1: Suppose $A \equiv p$. Then, we obtain the statement as follows.

$$a, \xi \Vdash_{\mathscr{X}} p \iff a \in \xi(p) \iff \upsilon(a) \in (\upsilon \circ \xi)(p) \iff \upsilon(a) \in [\![p]\!]_{\upsilon \circ \xi}$$

Case 2: Suppose $A \equiv B \to C$. Then, we obtain the statement by Lemma 4.4, as follows.

$$\begin{aligned}
& a, \xi \Vdash_{\mathscr{X}} B \to C \\
& \iff \exists U \in \mathcal{O}_{\mathcal{A}} X \ (a \in U \ \ \& \ \forall b \in U \ (b, \xi \Vdash_{\mathscr{X}} B \implies b, \xi \Vdash_{\mathscr{X}} C)) \\
& \iff \exists U \in \mathcal{O}_{\mathcal{A}} X \ (\upsilon(a) \in \upsilon(U) \ \ \& \ \forall b \in U \ \ (\upsilon(b) \in [\![B]\!]_{\upsilon \circ \xi} \implies \upsilon(b) \in [\![C]\!]_{\upsilon \circ \xi})) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by induction hypothesis)} \\
& \iff \upsilon(a) \in \bigcup \{U \in \mathcal{O}_{\mathsf{pt}^*}(\mathbf{\Omega}^* X) \mid [\![B]\!]_{\upsilon \circ \xi} \cap U \subseteq [\![C]\!]_{\upsilon \circ \xi}\} \\
& \iff \upsilon(a) \in [\![B \to C]\!]_{\upsilon \circ \xi}
\end{aligned}$$

Case 3. Suppose $A \equiv \forall p.B$. Then, we obtain the statement by Lemma 4.4, as follows.

$$\begin{aligned}
& a, \xi \Vdash_{\mathscr{X}} \forall p.B \\
& \iff \exists U \in \mathcal{O}_{\mathcal{A}} X \ \ (a \in U \ \ \& \ \forall b \in U \ \ \forall V \in d(b) \quad b, \xi(p\!:\!V) \Vdash_{\mathscr{X}} B) \\
& \iff \exists U \in \mathcal{O}_{\mathcal{A}} X \ \ (\upsilon(a) \in \upsilon(U) \ \ \& \ \forall b \in U \ \ \forall V \in d(b) \ \ \upsilon(b) \in [\![B]\!]_{\upsilon \circ (\xi(p:V))}) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by induction hypothesis)} \\
& \iff \upsilon(a) \in \bigcup \{U \in \mathcal{O}_{\mathsf{pt}^*}(\mathbf{\Omega}^* X) \mid \forall F \in U \ \ \forall V \in (\upsilon \circ d \circ \upsilon^{-1})(F) \ \ F \in [\![B]\!]_{(\upsilon \circ \xi)(p:V)}\} \\
& \iff \upsilon(a) \in [\![\forall p.B]\!]_{\upsilon \circ \xi}
\end{aligned}$$

Case 4: Suppose $A \equiv \exists p.B$. Then, by Corollary 3.3, we can prove the statement as in Case 3.

(2) Suppose A, ξ and $\upsilon(a) \in \mathsf{pt}^*(\mathbf{\Omega}^* X)$ satisfy $\xi(\mathrm{FV}(A)) \subseteq (\upsilon \circ d \circ \upsilon^{-1})(\upsilon(a))$. Then we have $a \in X$ and $(\upsilon^{-1} \circ \xi)(\mathrm{FV}(A)) \subseteq d(a)$ in the model $\mathscr{X}$, the fullness of $\mathscr{X}$ implies the existence of open sets $U \in d(a)$ and $V \in \mathcal{O}_{\mathcal{A}} X$ such that $a \in V$ and

$$b \in U \iff b, \ \upsilon^{-1} \circ \xi \Vdash_{\mathscr{X}} A$$

for every $b \in V$. This enables us to have $\upsilon(U) \in (\upsilon \circ d \circ \upsilon^{-1})(\upsilon(a))$ and $\upsilon(V) \in \mathcal{O}_{\mathsf{pt}^*}(\mathbf{\Omega}^* X)$ in the model $\mathbf{\Omega}^* \mathscr{X}$, for which $\upsilon(a) \in \upsilon(V)$ and

$$\upsilon(b) \in \upsilon(U) \iff \upsilon(b) \in [\![A]\!]_{\xi}$$

hold for every $\upsilon(b) \in \upsilon(V)$ by (1). □

Furthermore, we are able to present a translation in the reverse direction for a more general framework of models. For every **cHey**-lattice model $\mathscr{L} = \langle L, \sqsubseteq, d \rangle$, we introduce a **Top**-neighborhood model by

$$\mathsf{pt}^* \mathscr{L} = \langle \mathsf{pt}^* L, \mathcal{O}_{\mathsf{pt}^*} L, d \rangle.$$

Analogously to the preceding translation, we know that the satisfiability and the fullness on $\mathscr{L}$ is also preserved in the model $\mathsf{pt}^* \mathscr{L}$. This fact implies the soundness of $\mathbf{NJ_2}$ with respect to **cHey**-lattice models.

Lemma 5.2. Suppose $\mathscr{L} = \langle L, \sqsubseteq, d \rangle$ is a **cHey**-lattice model. Then, we have the following:
(1) For every A, ξ and $F \in \mathsf{pt}^* L$, we have $F \in [\![A]\!]_{\xi}$ in $\mathscr{L}$ if and only if $F, \xi \Vdash_{\mathsf{pt}^* \mathscr{L}} A$.
(2) If $\mathscr{L}$ is full, then $\mathsf{pt}^* \mathscr{L}$ is also full.

Proof:

(1) It is given by induction on the structure of A. Omitting proofs of trivial cases, we verify some cases below.

Case 1: Suppose $A \equiv p$. Then, we obtain the statement as follows.

$$F \in [\![p]\!]_\xi \iff F \in \xi(p) \iff F, \xi \Vdash_{\mathsf{pt}^*\mathscr{L}} p$$

Case 2: Suppose $A \equiv B \to C$. Then, we obtain the statement as follows.

$$\begin{aligned} F \in [\![B \to C]\!]_\xi &\iff \exists U \in \mathcal{O}_{\mathsf{pt}^*} L \ \ (F \in U \ \ \& \ \ [\![B]\!]_\xi \cap U \subseteq [\![C]\!]_\xi) \\ &\iff \exists U \in \mathcal{O}_{\mathsf{pt}^*} L \ \ (F \in U \ \ \& \ \ \forall G \in U \ \ (G \in [\![B]\!]_\xi \Longrightarrow G \in [\![C]\!]_\xi)) \\ &\iff \exists U \in \mathcal{O}_{\mathsf{pt}^*} L \ \ (F \in U \ \ \& \ \ \forall G \in U \ \ (G, \xi \Vdash_{\mathsf{pt}^*\mathscr{L}} B \Longrightarrow G, \xi \Vdash_{\mathsf{pt}^*\mathscr{L}} C)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by induction hypothesis)} \\ &\iff F, \xi \Vdash_{\mathsf{pt}^*\mathscr{L}} B \to C \end{aligned}$$

Case 3: Suppose $A \equiv \forall p.B$. Then, we obtain the statement as follows.

$$\begin{aligned} F \in [\![\forall p.B]\!]_\xi &\iff \exists U \in \mathcal{O}_{\mathsf{pt}^*} L \ \ (F \in U \ \ \& \ \ \forall G \in U \ \ \forall V \in d(G) \ \ G \in [\![B]\!]_{\xi(p:V)}) \\ &\iff \exists U \in \mathcal{O}_{\mathsf{pt}^*} L \ \ (F \in U \ \ \& \ \ \forall G \in U \ \ \forall V \in d(G) \ \ G, \, \xi(p:V) \Vdash_{\mathsf{pt}^*\mathscr{L}} B) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by induction hypothesis)} \\ &\iff F, \xi \Vdash_{\mathsf{pt}^*\mathscr{L}} \forall p.B \end{aligned}$$

Case 4: Suppose $A \equiv \exists p.B$. Then, by Corollary 3.3, we can prove the statement as in Case 3.

(2) It is immediate from (1). □

Theorem 5.3. Suppose $\mathbf{C}$ is a category such that $\mathbf{CDA} \subseteq \mathbf{C} \subseteq \mathbf{cHey}$. Then we have $\Gamma \vdash_{\mathbf{NJ}_2} A$ if and only if $\Gamma \models_{\mathbf{C}} A$.

Proof:

To see the "if" part, we consider Sobolev's canonical Kripke model $\mathscr{S} = \langle S, \leq, d \rangle$. This is regarded as an instance of full $\mathbf{Alex}$-neighbourhood models and, applying the transformation $\mathbf{\Omega}^*$ to $\mathscr{S}$, we obtain a $\mathbf{CDA}$-lattice model $\mathbf{\Omega}^*\mathscr{S}$. By Lemma 5.1 (2), the model $\mathbf{\Omega}^*\mathscr{S}$ is ensured to be full and we have $\Gamma \models_{\mathbf{\Omega}^*\mathscr{S}} A$ by the assumption. Now, let us suppose $a, \xi \Vdash_{\mathscr{S}} \Gamma$ for $a \in S$ and ξ such that $\xi(\mathrm{FV}(\Gamma, A)) \subseteq d(a)$. Then, in the model $\mathbf{\Omega}^*\mathscr{S}$, we have $(\upsilon \circ \xi)(\mathrm{FV}(\Gamma, A)) \subseteq (\upsilon \circ d \circ \upsilon^{-1})(\upsilon(a))$ and $\upsilon(a) \in [\![\Gamma]\!]_{\upsilon\circ\xi}$ by Lemma 5.1. Thus, we have $\upsilon(a) \in [\![A]\!]_{\upsilon\circ\xi}$ in $\mathbf{\Omega}^*\mathscr{S}$, from which $a, \xi \Vdash_{\mathscr{S}} A$ follows by Lemma 5.1 (1). This, together with Sobolev's completeness theorem [3, 4], implies $\Gamma \vdash_{\mathbf{NJ}_2} A$.

To see the "only-if" part, we suppose $\Gamma \vdash_{\mathbf{NJ}_2} A$. In a full $\mathbf{C}$-lattice model $\mathscr{L} = \langle L, \sqsubseteq, d \rangle$, we also suppose $F \in [\![\Gamma]\!]_\xi$ for $F \in \mathsf{pt}^* L$ and an environment ξ such that $\xi(\mathrm{FV}(\Gamma, A)) \subseteq d(F)$. By Lemma 5.2, we then obtain $F, \xi \Vdash_{\mathsf{pt}^*\mathscr{L}} \Gamma$ in the full $\mathbf{Top}$-neighbourhood model $\mathsf{pt}^*\mathscr{L} = \langle \mathsf{pt}^* L, \, \mathcal{O}_{\mathsf{pt}^*} L, \, d \rangle$. Thus Theorem 3.5 allows us to have $F, \xi \Vdash_{\mathsf{pt}^*\mathscr{L}} A$, from which $F \in [\![A]\!]_\xi$ holds in $\mathscr{L}$ by Lemma 5.2. So we conclude $\Gamma \models_{\mathbf{C}} A$. □

6. Concluding remarks

In this paper, we present a denotation of each proposition of $\mathbf{NJ_2}$ as an element of $\mathcal{O}_{\mathsf{pt}^*} L$ for a completely distributive algebraic lattice L, for which it would be also possible to present a denotation more directly as an element of L. This is because of the existence of an order isomorphism between $\mathcal{O}_{\mathsf{pt}^*} L$ and L ensured by the representation theorem. As for this semantical framework, one of future problems is to clarify the scope of its application. Indeed, we have various extensions of the syntax of $\mathbf{NJ_2}$ formalizing inference rules of higher-order logic, for which we do not present enough observation at this stage about whether our framework works as a model or not. For example, if we intend to adjust our result to the semantics of higher-order propositional logic, then it might be natural to demand the structure of cartesian closed category in addition. However, on the other hand, the category $\mathbf{Alg}$ is known not to be cartesian closed. So, taking the arrow part into account, we possibly need to confine ourselves to a subcategory of $\mathbf{Alg}$ and to establish a Stone duality based on this restricted viewpoint.

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